The Algebra of complex numbers:

Negative numbers $\longleftrightarrow$ $x+1=0$ $(x=-1)$

Rational numbers $\longleftrightarrow$ $2x+3=0$ $(x=-3/2)$

Irrational numbers $\longleftrightarrow$ $x^2-2=0$ $(x=\pm\sqrt{2})$

Complex numbers also arise via this process.

Consider the eqn

$$x^2+1=0$$ (Just as we did $-1, \sqrt{2}\ldots$)

Introduce a solution of the above $\int$. Call it i.

$$i^2=-1$$

Solns of $x^2+1=0$ are $\pm i$.

$$i=\sqrt{-1}.$$

Notations:

$\mathbb{R}$ = set of real numbers.

The set of complex numbers, denoted $\mathbb{C}$, is the set

$$\mathbb{C}=\{a+ib \mid a, b\in\mathbb{R}\}$$

- $a+ib = c+id$, $a,b,c,d\in\mathbb{R}$ if and only if $a=c$ and $b=d$.
- Real numbers can be regarded as elements

of $\mathbb{C}$. For example, if $a \in \mathbb{R}$, then we regard a as the complex number $a + i \cdot 0$. In this sense $\mathbb{R} \subset \mathbb{C}$.

- $a + ib = 0$ if and only if $a = 0$ and $b = 0$.
- By $a - ib$, $a, b \in \mathbb{R}$, we mean the complex number $a + i(-b)$.
- If $c \in \mathbb{R}$, $c \neq 0$, then $\frac{a+ib}{c} := \frac{a}{c} + i\frac{b}{c}$, $a, b \in \mathbb{R}$.

If $a + ib \neq 0$ then either a or b is non-zero. In particular $a^2 + b^2 \neq 0$.

Arithmetic operations on complex numbers: $a, b, c, d \in \mathbb{R}$.

1. $(a+ib) + (c+id) = a + c + i(b+d)$
2.
$$\begin{aligned}(a+ib)(c+id) &= a(c+id) + ib(c+id)\\ &= ac + iad + ibc + i^2 bd\\ &= ac + i(ad+bc) - bd\\ &= (ac - bd) + i(ad+bc)\end{aligned}$$

Consequence: $(c+id)(c-id) = c^2 + d^2 + i(-cd + cd)$
$= c^2 + d^2$.

$$\boxed{(c+id)(c-id) = c^2 + d^2}$$

In particular, if $c+id \neq 0$ then

$$(c+id)(c-id) = c^2+d^2 \neq 0$$

3. <u>Division</u>: Suppose $a, b, c, d \in \mathbb{R}$ and suppose $c+id \neq 0$. We'd like to divide $a+ib$ by $c+id$.

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}$$

The Real and Imaginary parts of a complex number:

Let $z = a+ib \in \mathbb{C}$, $a, b \in \mathbb{R}$.

$Re(z) = a$ = the "real" part of z

$Im(z) = b$ = the "imaginary" part of z.

Examples:

1. Let $z_1 = 3+i$, $z_2 = 2-4i$

$$z_1+z_2 = (3+2)+(1-4)i = 5-3i.$$

$$z_1 z_2 = (3+i)(2-4i)$$

$$= (6-(-4)) + i(2-12)$$

$$= 10-10i$$

$$\frac{z_1}{z_2} = \frac{3+i}{2-4i} = \frac{3+i}{2-4i}\cdot\frac{2+4i}{2+4i} = \frac{(3+i)(2+4i)}{2^2+4^2}$$

$$= \frac{3(2+4i)+i(2+4i)}{20} = \frac{6+12i+2i-4}{20}$$

$$= \frac{2+14i}{20}$$

$$= \frac{1+7i}{10}$$

$$= \frac{1}{10} + \frac{7}{10}i$$

2. Compute i^{2023}.

Soln: $i, i^2, i^3, i^4, i^5, i^6, i^7, i^8, \ldots$

$i, -1, -i, 1, i, -1, -i, 1, \ldots$

Note

$$2023 = 4(505) + 3$$

So

$$i^{2023} = i^{4(505)+3} = (i^4)^{505} \cdot i^3$$

$$= 1^{505} \cdot (-i)$$

$$= -i.$$

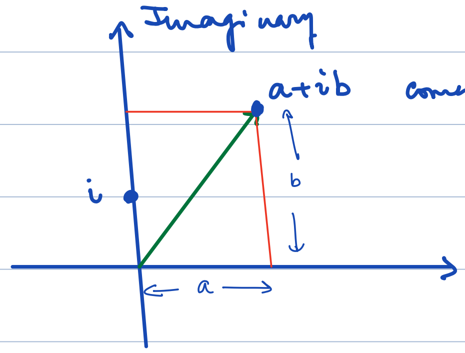
Definition: Let $z = a+ib \in \mathbb{C}$, $(a, b \in \mathbb{R})$. The complex conjugate of z is the number $a-ib$. This is denoted $\bar{z}$.

$$\bar{z} = a - ib \qquad (\text{if } z = a+ib,\ a, b \in \mathbb{R})$$

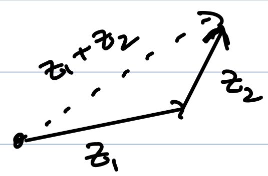
Note $\boxed{z \cdot \bar{z} = a^2 + b^2.}$

Point representation of complex numbers (Due to ARGAND)

Let $z \in \mathbb{C}$, $Re(z) = a$, $Im(z) = b$.



Addition and subtraction — as in 2-D vectors.



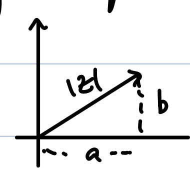
Vector representation of addition.

Definition (Modulus or absolute value): Let $z \in \mathbb{C}$, $Re(z) = a$, $Im(z) = b$, $(z = a + ib)$. The modulus (or abs. value) of z is:

$$|z| = \sqrt{a^2 + b^2}$$

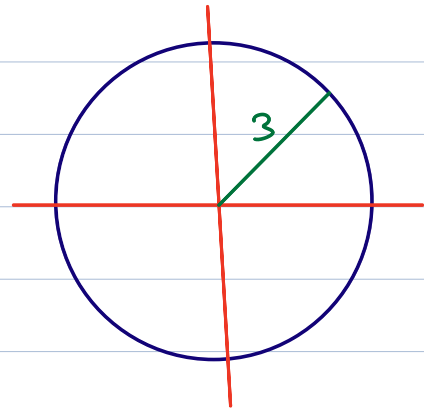
So

- $|z|$ is the distance of the point representation of z from the origin.
- The length of the "vector" z



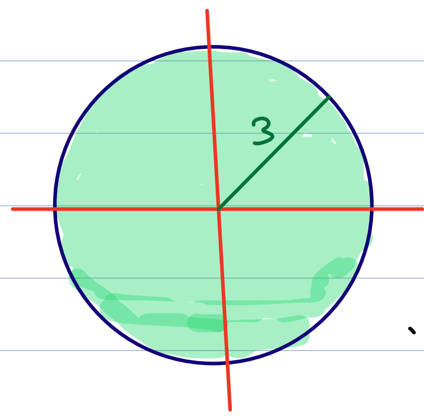
Examples:

1. Describe the set of points z in the complex plane such that $|z| = 3$.



Ans: The circle of radius 3 centred at 0.

2. Same as above but with $|z| \leq 3$.



Ans: The closed disc of radius 3 centred at 0.

3. Describe the set of complex numbers in the complex plane such that

$$|z - 1 + 2i| = |z + 3 - i|$$

$|z - (1-2i)|$

Solution:

Let $z = a + ib$, $a, b \in \mathbb{R}$.

$$|a + ib - 1 + 2i| = |a + ib + 3 - i|$$

i.e. $\quad |(a-1)+i(b+2)| = |(a+3)+i(b-1)|$

So

$$(a-1)^2 + (b+2)^2 = (a+3)^2 + (b-1)^2$$

$$\cancel{a^2} - 2a + 1 + \cancel{b^2} + 4b + 4 = \cancel{a^2} + 6a + 9 + \cancel{b^2} - 2b + 1$$

$$-2a + 4b + 5 = 6a - 2b + 10$$

$$8a - 6b = -5.$$