Name: ——————

QUIZ 1

Recall the Cauchy estimates

$$|f^{(n)}(a)| \leq n!\frac{M}{R^n} \quad (n = 1, 2, 3, \ldots)$$

for an analytic function on the closed disc $\overline{B}(a,\, R)$ where M is an upper bound for $|f(\zeta)|$ on the circle C_R of radius R around a.

A function is said to be *entire* if it is defined and analytic on the entire complex plane.

(1) Show using the Cauchy estimates that a bounded entire function is necessarily constant. (This is called *Louisville's theorem*)

Solution. Let f be an entire function which is bounded. Let $M < \infty$ be an upper bound for $|f|$ on $\mathbf{C}$. Let $a \in C$. Since f is analytic on $\overline{B}(a,\, R)$ for every $R > 0$, for each $R > 0$ the Cauchy estimates give

$$|f'(a)| \leq \frac{M}{R}.$$

Letting $R \longrightarrow \infty$ we see that $f'(a) = 0$. Since a is arbitrary, f is a constant.

There is a slightly different way of doing the same problem. By Cauchy estimates for f on $\overline{B}(a, R)$ we get for $n \geq 0$

$$|f^{(n)}(0)| \leq n!\frac{M}{R^n}.$$

For $n \geq 1$, if we let $R \longrightarrow \infty$, the right side tends to zero. It follows that

$$f^{(n)}(0) = 0 \qquad (n \geq 1).$$

Using the power series expansion of f around 0 we get (z) is a constant. □

(2) Let a be a point of a region Ω and f an analytic function on $\Omega \smallsetminus \{a\}$ such that $(z-a)f(z) \to 0$ as $z \to a$. Show that f can be extended to an analytic function on Ω. (This is called *Riemann's removable singularities theorem*). [Hint: Use a homework problem you did involving an integral formula.]

Solution. Let $B = B(b, r)$ be an open ball in Ω such that $a \in B$ and the closed ball $\overline{B}$ is contained in Ω. Let C be the boundary circle of $\overline{B}$. Then from Problem 5 from HW 1, or from Theorem 1.1 from Lecture 2, we see that

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{\zeta - z}$$

is an analytic function on B. On the other hand, by Cauchy's theorem we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{\zeta - z} \qquad (z \in B \smallsetminus \{a\})$$

since for fixed $z \in B \smallsetminus \{a\}$, the function $h\colon B \smallsetminus \{z, a\} \to \mathbf{C}$ defined by the formula

$$h(w) = \frac{f(w) - f(z)}{w - z} \qquad (w \in B \smallsetminus \{z, a\})$$

is analytic and $\lim_{w \to w_0} (w - w_o)h(w) \longrightarrow 0$ for $w_0 \in \{a, z\}$. It follows that g extends f on B. This means f can be extended to an analytic function on Ω.