QUIZ 4

(1) Suppose Ω is a simply connected region and u a harmonic function on Ω. Show that u has a harmonic conjugate on Ω. You may use the fact that on a simply connected region every C^1 differential 1-form which is closed, is exact.

Recall that a function u on a region Ω is said to have *local averaging property* if for each $a \in \Omega$ there exists a positive number $\rho(a)$ such that the closed disc $|z-a| \leq \rho(a)$ lies in Ω and

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta \qquad (r \leq \rho(a)).$$

(2) Let $a \in \mathbf{R}$ (where $\mathbf{R}$ is the set of real numbers). Let $D = B(a, r)$, where $r > 0$. Let $D^+ = \{z \in D \mid \mathrm{Im}(z) > 0\}$, $D^- = \{z \in D \mid \mathrm{Im}(z) < 0\}$, and $L = D \cap \mathbf{R}$. Let v be a harmonic function on D^+ such that for $x \in L$ we have $v(z_n) \to 0$ as $n \to \infty$ for every sequence of points $\{z_n\}$ in D^+ which converges to x. Let v^* be defined by

$$v^*(z) = \begin{cases} v(z) & \text{if } z \in D^+ \\ 0 & \text{if } z \in L \\ -v(\bar{z}) & \text{if } z \in D^- \end{cases}$$

(a) Show that v^* has the local averaging property on D.

(b) Assume that continuous functions with local averaging property are harmonic. Show there exists a holomorphic function f on D such that $\mathrm{Im}(z) = v^*(z)$ for $z \in D$, and that such functions are unique up to addition by a real constant.

(c) Show, without using the Schwarz Reflection Principle, that that for f as in (b), we have $\overline{f(\bar{z})} = f(z)$ for $z \in D$.