Name: ______________

QUIZ 1 (MAKE UP)

Recall *Riemann's removable singularities theorem* which was given as a problem in the original quiz (for which this is a make up). It states that if Ω is a region, $a \in \Omega$ a point, and f an analytic function on $\Omega \setminus \{a\}$ such that $\lim_{z\to a}(z-a)f(z) = 0$, then f can be extended to an analytic function on all of Ω.

(1) Suppose $f(z)$ is analytic on $U = B(0,r) \setminus \{0\}$ and $\lim_{n\to\infty} z^n f(z) \neq 0$ for any $n \geq 1$ (for each $n \geq 0$ this means that either the limit does not exist, or if it does, it is not zero). Show that $f(U)$ is dense in $\mathbf{C}$. [Hint: Assume $f(U)$ is not dense, i.e., its complement in $\mathbf{C}$ contains an open set. Derive a contradiction.] (This is knows as the *Weierstrass-Casorati Theorem.*)

(2) Suppose $f(z)$ is analytic on a region Ω and $|f(z)|$ has a maximum at $a \in \mathbf{C}$. Show that $f(z)$ is a constant using the Inverse Function Theorem from Several Variables Calculus. [Hint: You may need a fact proven in class today regarding the function $|f|$ on Ω. You may also need Riemman's removable singularities theorem.]