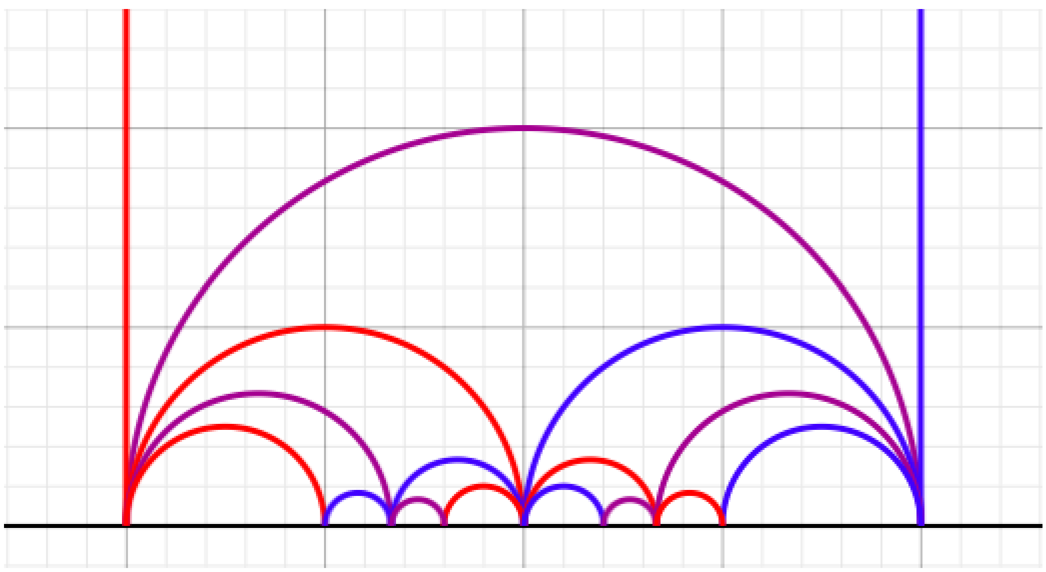
RECREATIONAL PROBLEM IN FINAL EXAM

Here are four generations of reflections of the original three semicircles namely the large purple semi-circle of radius 1/2 centred at 1/2 and the two circles of radius 1/4 coloured red (on the left) and blue (on the right).



You will notice, by visual inspection, that for each generation of semicircles, the ones on the extreme left and extreme right are the largest and have diameter $1/n$ where n=generation. The number $1/n$ comes from the formula in the exam, namely

$$d_n = \frac{rs}{(n-1)s-(n-2)r}$$

with $s = 1$ and $r = 1/2$. See if you can actually prove an n-th generation semi-circle has diameter $\leq 1/n$. You will have to use the fact that the formula for d_n is an increasing function of r for fixed s and n. Now take a typical generation n semi-circle. There is a history of reflections from the original purple semi-circle and one of its red or blue subservient semi-cricles, all the way to your given semi-circle:

$$S_1 \to S_2 \to \cdots \to S_n.$$

S_1 is the purple semi-circle with radius 1/2. S_2 is one of the two semi-circles of radius 1/4. WLOG say it is the red one. If all the reflections after that end up on the left (the dominant reflection) then S_n has diameter $1/n$. What happens if there is reflection of a non-dominant sort (the reflection of the smaller arc in the two new arcs of the $n-1$ generation triangle)? Use the monotone nature of d_n mentioned above (for r, with s and n fixed).

Let $T_{i,1}, T_{i,2}, \ldots, T_{i,2^{i-1}}$ be all the i^{th} generation triangles (by a traingle we mean the three arcs, the interior, but not the vertices).

Let T_0 denote the area bounded by and including the two vertical lines and the large purple semi-circle, but omitting the "vertices" 0 and 1. Let

$$R = \{z \mid \Im(z) > 0,\, 0 \leq \Re(z) \leq 1\}$$

and

$$R_n = \{z \mid \Im(z) > 1/2n,\, 0 \leq \Re(z) \leq 1\}.$$

Then $\bigcup_n R_n = R$. The assertion that the n^{th} generation semi-circle has diameter $\leq 1/n$ means that R_n is contained in the union

$$A_n = T_0 \cup \bigcup_{i \leq n} \bigcup_{j=1}^{2^{i-1}} T_{ij}.$$

This in turn shows that $R = \bigcup_n R_n \subset \bigcup_n A_n \subset R$, i.e.,

$$R = T_0 \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} T_{ij}.$$

The area T_0 can be regarded as "triangle" on the Riemann sphere, in other words as a "0^{th} generation triangle".

The terminology I have used is deliberately informal (for example what I called a triangle is not a standard triangle, especially since vertices are removed and interiors included). That is because this is not really a solution but a strategy fora solution.

Anyway, have fun filing in the details and working out the problem. Formulate and set yourself a similar problem associated to the other related proof of Picard's Little Theorem that was given in class (the one in Lecture 26).