SOLUTIONS TO SELECTED PROBLEMS

Problem 2, HW 5. We have to show that if $\varphi\colon \Omega \times I \to \mathbf{C}$ is a continuous function where $I = [\alpha, \beta]$ is a closed interval in $\mathbf{R}$ and Ω is a region in $\mathbf{C}$, and if for each fixed $t \in I$, the function $z \mapsto \varphi(z, t)$ is holomorphic on Ω then

$$F(z) := \int_\alpha^\beta \varphi(z, t)dt$$

is a holomorphic on Ω and $F'(z) = \int_\alpha^\beta (\partial\varphi(z, t)/\partial z)dt$. Fix $t \in I$. Let D be an open disc in Ω such that the closed disc $\overline{D}$ lies in Ω and let C be the bounding circle for these discs. By Cauchy's integral formula and by the hypotheses, we have

$$\varphi(z, t) = \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta,\, t)}{\zeta - z}d\zeta \quad (z \in D).$$

Therefore for $z \in D$ we have

$$F(z) = \frac{1}{2\pi i}\int_I \int_C \frac{\varphi(\zeta,\, t)}{\zeta - z}d\zeta dt.$$

Now $K = C \times I$ is compact and hence φ is bounded on K. Let this bound be M. Next, if $z \in D$, then the distance between z and C is positive, say it is ρ. Then $|\zeta - z| > \rho$ for all $\zeta \in C$. Thus for this fixed z we have

$$\left|\frac{\varphi(z,\, t)}{\zeta - z}\right| \leq \frac{M}{\rho} \quad \text{on } K.$$

Fubini's theorem applies and therefore have

$$\begin{aligned} F(z) &= \int_C \left(\frac{1}{2\pi i}\int_\alpha^\beta \varphi(\zeta,\, t)dt\right)\frac{d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta - z}d\zeta \end{aligned}$$

Now $F(\zeta)$ is clearly continuous as a function of the variable ζ on C. Indeed, since K is compact, φ is uniformly continuous on K, and hence given $\epsilon > 0$ we can find $\delta > 0$ such that $|\varphi(\zeta_1, t_1) - \varphi(\zeta_2, t_2)| < \epsilon$ whenever the distance between (ζ_1, t_1) and (ζ_2, t_2) is less than δ in $K \subset \mathbf{C} \times \mathbf{R} \subset \mathbf{R}^3$. In particlular, if $|\zeta_1 - \zeta_2| < \delta$ then $|\varphi(\zeta_1, t) - \varphi(\zeta_2, t)| < \epsilon$ for all $t \in I$. It is then easy to see that $|F(\zeta_1) - F(\zeta_2)| < \epsilon(\beta - \alpha)$, giving the required continuity. In particular F is measurable (this can be seen by other methods too). By Theorem 1.1 of Lecture 2, or by problem 5 of HW 1, we see that $F(z)$ is holomorphic in D. It follows that $F(z)$ is holomorphic on Ω. Again by either Theorem1.1 of Lecture 2, or by problem 5 of HW 1, we have

$$F'(z) = \int_C \left(\frac{1}{2\pi i}\int_\alpha^\beta \varphi(\zeta, t)dt\right)\frac{d\zeta}{(\zeta - z)^2}.$$

Apply Fubini again (same argument holds, but now the bound for the integrand is M/ρ^2) and conclude that

$$\begin{aligned} F'(z) &= \int_\alpha^\beta \left(\frac{1}{2\pi i} \int_C \frac{\varphi(\zeta,t)}{(\zeta - z)^2} d\zeta \right) dt \\ &= \int_\alpha^\beta \frac{\partial}{\partial z} \varphi(z,t) dt \end{aligned}$$

where yet once again we are using problem 5 from HW 1 etc.

Problem 6, HW 6. Note that since $f'(a) \neq 0$ the order of the zero of f at a is 1. From the hypothesis that a is the only solution of $f(z) = 0$ in $\overline{D}$ it follows that $f(z) = (z-a)h(z)$ with h holomorphic and nowhere vanishing on $\overline{D}$. In particular on $\overline{D}$ we have

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a} + \frac{h'(z)}{h(z)}$$

and

$$\frac{zf'(z)}{f(z)} = \frac{z}{z-a} + z\frac{h'(z)}{h(z)}.$$

Since $h(z)$ is nowhere vanishing on $\overline{D}$, the functions $h'(z)/g(z)$ and $zh'(z)/g(z)$ are holomorphic there whence their integrals over C vanish. We then have the following sequence of equalities: $(1/2\pi i)\int_C zf'(z)/f(z)dz = (1/2\pi i)\int_C z/(z-a)dz =$a. (The last equality can be deduced, for example, from Cauchy's integral formula.) This proves part (a).

Next let $\Gamma = f(C)$. Let W be the connected component of $\mathbf{C} \setminus \Gamma^*$ which contains 0 (recall f is never zero on C by our hypotheses). We have seen that

$$\eta(\Gamma, 0) = \frac{1}{2\pi i}\int_\Gamma \frac{dw^*}{w^*} = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_C \frac{1}{z-a}dz = 1.$$

Hence $\eta(\Gamma, w) = 1$ for all $w \in W$. This means $(1/2\pi i)\int_C f'(\zeta)/(f(\zeta) - w)d\zeta = 1$ for all $w \in W$, i.e., that $f(z) - w$ has only one zero, call it $g(w)$, in $\overline{D}$, for every $w \in W$. Thus $f(z) - w = (z - g(w))r(z)$ on $\overline{D}$, where $r(z)$ is holomorphic and nowhere vanishing on $\overline{D}$. As before (noting that $r'(z)/r(z)$ and $zr'(z)/r(z)$ are holomorphic on $\overline{D}$ and hence have vanishing integrals over C), we conclude that

$$g(w) = (1/2\pi i)\int_C \zeta \frac{f'(\zeta)}{f(\zeta) - w} d\zeta.$$

Since $z = g(w)$ is the only solution of $f(z) - w = 0$ in $\overline{D}$, it follows that $f(g(w)) = w$. Similarly if $\zeta \in g(W)$, then we have $w = f(\zeta) \in W$, whence the equation

$$f(z) - w = 0$$

has only one solution in $\overline{D}$. Since $z = \zeta$ and $z = g(w)$ are both solutions in $\overline{D}$ of the above equation, it follows that $g(w) = \zeta$, i.e. $g(f(\zeta)) = \zeta$. This finishes part (b).

Now for part (c). Let $t \mapsto \zeta(t)$, $0 \le t \le 2\pi$ be the usual parametrisation on C (i.e., $\zeta(t) = a + re^{it}$ for $0 \le t \le 2\pi$). Then setting

$$\varphi(w,t) = \frac{1}{2\pi i}\zeta(t)\frac{f'(\zeta(t))}{f(\zeta(t)) - w}\zeta'(t)$$

and using Problem 2 of HW 5 (see solution above), we conclude that $g(w)$ is holomorphic on W.