# LECTURE 9

**Date of Lecture:** February 2, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Some results were unfortunately not stated in class (see Proposition 1.1.1 below). Others are a quick summary.

## 1. Open Mapping Theorem

There are many proofs of the open mapping theorem. Here is a favourite.

**Theorem 1.1.** *Let $f\colon \Omega \to \mathbf{C}$ be a non-constant analytic function on a region $\Omega$. The $f$ is an open map, i.e., $f(U)$ is an open set in $\mathbf{C}$ whenever $U$ is open in $\Omega$.*

*Proof.* Let $w_0 \in \mathbf{C}$ be in $f(\Omega)$. We have to show that there is an open neighbourhood $V$ of $w_0$ such that $V \subset f(\Omega)$. To that end, pick a pre-image $z_0 \in \Omega$ of $w_0$. Let $\delta > 0$ be so small that $\overline{B}(z_0,\, \delta) \subset \Omega$ and $z_0$ is the only solution of $f(z) = w_0$ in $\overline{B}(z_0,\, \delta)$. Such a $\delta$ exists since $f(z) - w_0$ is non-constant, whence its zeros are isolated. Let $C$ be the bounding circle $\{|z - z_0| = \delta\}$ oriented in the usual way, via the parameterisation $t \mapsto z_0 + \delta e^{it}$, $0 \le t \le 2\pi$. Let $n$ be the order of the order of the zero of $f(z) - w_0$ at $z_0$. Note that $n \ge 1$ since $f(z_0) = w_0$. From the Argument Principle we have

$$(*) \qquad \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z) - w_0}dz = n.$$

Let $\Gamma$ be the the image path $f(C)$, i.e., $\Gamma$ is the path given by $t \mapsto f(z_0 + \delta e^{it})$, $0 \le t \le 2\pi$. Then $(*)$ can be re-written as $\frac{1}{2\pi i}\int_\Gamma (w - w_0)^{-1}dw = n$, i.e.,

$$\eta(\Gamma, w_0) = n.$$

Let $V$ be the connected component of $\mathbf{C} \smallsetminus \Gamma^*$ containing $w_0$. Then, since the winding number is constant on connected components of $\mathbf{C} \smallsetminus \Gamma^*$, we get

$$\eta(\Gamma, w) = n \qquad (w \in V).$$

It follows that $\frac{1}{2\pi i}\int_C f'(z)/(f(z) - w)dz = n$ for $w \in V$. By the Argument Principle, this means $f(z) - w = 0$ has $n$ solutions in $B(z_0,\, \delta)$ (counted with multiplicity) for every $w \in V$. Since $n \ge 1$, this shows that $V \subset f(\Omega)$, which is what we were required to show. $\square$

Note that the method of the proof also proves most of the following (compare with [A, p.131, Thm 11]).

**Proposition 1.1.1.** *Suppose that $f(z)$ is analytic at $z_0$, $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order $n$ at $z_0$. If $\varepsilon > 0$ is sufficiently small, there exists a $\delta > 0$ such that for all $a$ with $0 < |a - w_0| < \delta$ the equation $f(z) = a$ has exactly $n$ roots in the disc $|z - z_0| < \varepsilon$.*

*Proof.* The only issue is the phrase "exactly $n$ roots" in the statement. This is taken to mean the $n$ roots of the equation $f(z) = w$ are distinct for $w$ in the punctured disc $0 < |a - w_0| < \delta$. This is easily achieved if the positive number $\delta$ in the proof of Theorem 1.1 is taken to be so small that in the closed disc of radius $\delta$ centred at $z_0$ we have $f'(z) \neq 0$ if $z \neq z_0$. Since $f(z)$ is non-constant (otherwise the order of $f(z) - w_0$ at $z_0$ does not make sense), $f'(z)$ cannot vanish identically. Its zeros, if they exist, are isolated making the choice of such a $\delta$ possible. □

## 2. **Harmonic Functions**

**2.1.** A $C^2$ function $u$ on an open set $W$ in $\mathbf{R}^n$ is said to be *harmonic* if

$$\Delta u(\boldsymbol{x}) := \frac{\partial^2 u}{\partial x_1^2}(\boldsymbol{x}) + \cdots + \frac{\partial^2 u}{\partial x_n^2}(\boldsymbol{x}) = 0 \qquad (\boldsymbol{x} \in W),$$

where $x_i$, $i = 1, \ldots, n$ are the standard coordinate functions on $\mathbf{R}^n$. It turns out that harmonic functions are necessarily $C^\infty$.

Suppose $\boldsymbol{x} \in W$ and $B$ is a ball centred at $\boldsymbol{x}$ whose closure lies in $W$. Let $S = \partial B$ the corresponding bounding sphere. Let $V$ denote the Lebesgue measure on $\mathbf{R}^n$ and $A$ the "surface area" measure on $S$. It turns out that harmonic functions $u$ have the following two equivalent averaging properties:

$$(A_1) \qquad u(\boldsymbol{x}) = \frac{1}{V(B)} \int_B u dV$$

and

$$(A_2) \qquad u(\boldsymbol{x}) = \frac{1}{A(S)} \int_S u dS.$$

Conversely, it turns out that if $u$ is continuous on $W$ and has the averaging property then $u$ is harmonic.

We will not be proving any of these results for general $n$. However for $n = 2$ we will prove the averaging property for harmonic functions. This is essentially Cauchy's integral formula for the centre of a disc, as we will see. The connection with our course is that the real and imaginary parts of an analytic function are harmonic.

**Theorem 2.1.1.** *Let $f = u + iv$ be an analytic function on a region $\Omega$ with $u$ and $v$ the real and imaginary parts of $f$. Then $u$ and $v$ are harmonic.*

*Proof.* We know that $u$ and $v$ are $C^\infty$. In particular it is $C^2$ Now $f' = u_x + iv_x$. Applying Cauchy-Riemann to $f'$ we see that

$$(*) \qquad u_{xx} = v_{xy}$$

and

$$(**) \qquad u_{xy} = -v_{xx}.$$

By Cauchy-Riemann for $f$, we have $v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}$. Substituting in $(*)$ we get $u_{xx} + u_{yy} = 0$. Similarly $u_{xy} = v_{yy}$ and substituting in $(**)$ we get $v_{xx} + v_{yy} = 0$. □

**2.2. Harmonic Conjugates.** Let $u$ be a harmonic function on a region $\Omega$ in $\mathbf{C}$. A *harmonic conjugate* of $u$ is a function $v$ on $\Omega$ such that $u + iv$ is analytic on $\Omega$. Note that a harmonic conjugate of $u$ is necessarily harmonic by Theorem 2.1.1. If $v$ and $w$ are harmonic conjugates of $u$ then by Cauchy-Riemann, $v_x = w_x$ and $v_y = w_y$ whence $v - w$ is constant on $\Omega$. Thus harmonic conjugates are unique up to a real constant.

**Theorem 2.2.1.** *If $u$ is harmonic on disc $D$ in $\mathbf{C}$ then it has a harmonic conjugate.*

*Remark:* Instead of a disc we can substitute any simply connected region as the general Cauchy theorem will show when we prove it. Equally well one can use Green's Theorem, or for that matter De Rham's theorem to prove the statement for simply connected regions.

*Proof.* Let $U = u_x$, $V = -u_y$ and $g = U + iV$. Since $u$ is $C^2$ we have $u_{xy} = u_{yx}$ giving $U_y = -V_x$. Further, the condition $u_{xx} + u_{yy} = 0$ translates to $U_x = V_y$. Thus $U$ and $V$ satisfy the Cauchy-Riemann equations, whence $g$ is analytic. We have seen that on a disc analytic functions have primitives. Let $f$ be a primitive of $g$. Without loss of generality we may assume $f(z_0) = u(z_0)$ where $z_0$ is the centre of $D$. Suppose $f = \varphi + i\psi$ is the decomposition of $f$ into its real and imaginary parts. Note that since $f(z_0) = u(z_0) \in \mathbf{R}$, $\psi(z_0) = 0$ and $u(z_0) = \varphi(z_0)$. Now $g = f' = \varphi_x + i\psi_x = \varphi_x - i\varphi_y$. Thus $\varphi_x = u_x$ and $\varphi_y = u_y$. Since $u(z_0) = \varphi(z_0)$, we get $u = \varphi$. It follows that $\psi$ is a harmonic conjugate of $u$. □

**Remark 2.2.2.** Suppose $u$ and $D$ are as in the theorem. If one considers the differential $\omega = -u_y dx + u_x dy$ then one sees that $d\omega = (u_{xx} + u_{yy})dx \wedge dy = 0$. On $D$, or for that matter any simply connected region $\Omega$, which necessarily has its first cohomology equal to zero (for, given a base point $z_0 \in \Omega$, we have $H^1(\Omega, \mathbf{R}) \cong \mathrm{Hom}(\pi_1(\Omega, z_0), \mathbf{R}) = 0$), De Rham's theorem gives us the existence of a $C^2$ function $v$ such that $dv = \omega$. It is easy to see that $v$ is a harmonic conjugate of $u$. Equivalently, $v$ is obtained by setting $v(z) = \int_{\gamma_z} \omega$ where $\gamma_z$ is any path starting at $z_0$ and ending at $z$. The path independence is assured by Green's theorem, which is Stokes' theorem on the plane.

## References


[A] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New-York, 1979.