## LECTURE 8

**Date of Lecture:** February 1, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

### 1. Isolated Singularities

**1.1.** We begin with the following definition.

**Definition 1.1.1.** An analytic function $f(z)$ is said to have an *isolated singularity at* $a \in \mathbf{C}$ if the domain of $f(z)$ contains $B(a,\, r) \smallsetminus \{a\}$ for some $r > 0$.

Suppose $f(z)$ has an isolated singularity at $a$. Fix a punctured disc $B^* = B(a,\, r) \smallsetminus \{a\}$ contained in the domain of $f(z)$. If $f|_{B^*}$ can be extended as an analytic function to $B = B(a, r)$ we say $f(z)$ has a *removable singularity at* $a$. We say $f(z)$ has a *pole at* $a$ if $\lim_{z\to a} f(z) = \infty$. We say $f(z)$ has an *essential singularity* if the isolated singularity at $a$ is neither removable nor a pole.

**Remark 1.1.2.** Suppose $f(z)$ has an isolated singularity at $z = a$. Recall (from Problem 2 of Quiz 1) that if $\lim_{z\to a}(z-a)f(z) = 0$ then $f(z)$ has a removable singularity at $z = a$ (Riemann's removable singularities theorem).

**Proposition 1.1.3.** *Let $\Omega$ be a region, $a \in \Omega$ a point in, and write $\Omega^* = \Omega \smallsetminus \{a\}$. Suppose $f\colon \Omega \smallsetminus \{a\} \to \mathbf{C}$ is an analytic function. Then $f$ has a pole at $z = a$ if and only if*

$$f(z) = \frac{\varphi(z)}{(z-a)^n} \qquad (z \in \Omega \smallsetminus \{a\})$$

*for some integer $n \geq 1$ and some analytic function $\varphi$ on $\Omega$ with $\varphi(a) \neq 0$. The above representation of $f(z)$ is unique, i.e., $\varphi$ and $n$ are unique, depending only on $f$.*

*Proof.* Let us first dispose off the uniqueness assertion. Suppose $f(z)$ can be represented as asserted. Then the positive integer $n$ is characterised by the property that $\lim_{z\to a}(z-a)^{n+j}f(z) = 0$ for $j \geq 1$ but $\lim_{z\to a}(z-a)^n f(z)$ exists in $\mathbf{C}$ and is non-zero. The uniqeness of $\varphi$ follows since $\varphi(z) = (z-a)^n f(z)$ on $\Omega^*$, which is dense in $\Omega$. Suppose $f(z)$ has a pole at $z = a$. Then there is a ball $B = B(a, r)$ in $\Omega$ such that $f(z)$ is nowhere vanishing on $B^* = B \cap \Omega^*$. (Indeed, since $f(z) \to \infty$ as $z \to a$, for every $M > 0$ there exists $\rho > 0$ such that $|f(z)| > M$ for $0 < |z-a| < \rho$.) Define $g\colon B^* \to \mathbf{C}$ by the formula

$$g(z) = \frac{1}{f(z)} \qquad (z \in B^*).$$

Then $\lim_{z\to a} g(z) = 0$. Thus by Riemann's removable singularities theorem, $g(z)$ can be regarded as an analytic function on $B$ with its value at $a$ being $g(a) = 0$. It follows that we have a unique decomposition $g(z) = (z-a)^n h(z)$ with $n \geq 1$ and

$h(z)$ analytic on $B$ with $h(a) \neq 0$. Moroever, for $z \in B^*$, $g(z) \neq 0$. Thus $h(z)$ is nowhere vanishing, and hence $\varphi(z) := h(z)^{-1}$ is analytic on $B$. Thus on $B$ we have

$$f(z) = \frac{\varphi(z)}{(z-a)^n},$$

and since $(z-a)^n f(z)$ is analytic on $\Omega^*$, $\varphi(z)$ extends to all of $\Omega$. It is clear that $\varphi(a) = h(a)^{-1} \neq 0$.

Conversely, suppose $f(z)$ has the asserted representation. Let $B = B(a, r)$ be a ball in $\Omega$ and set $B^* = B \cap \Omega^*$. On $B$, $\varphi(z)$ has a power series representation

$$\varphi(z) = \sum_{n=0}^{\infty} b_n (z-a)^n.$$

We have $b_0 \neq 0$ since $b_0 = \varphi(a) \neq 0$. Since $\lim_{z-a} \varphi(z) = b_0$, there exists $\delta > 0$ such that

$$|\varphi(z)| > \frac{|b_0|}{2} \qquad (|z-a| < \delta).$$

This yields

$$|f(z)| > \frac{|b_0|}{2|z-a|^n} \qquad (z \in B(a, \delta) \setminus \{a\}).$$

It follows that $\lim_{z-a} f(z) = \infty$, i.e., $f(z)$ has a pole at $z = a$. □

**Definition 1.1.4.** If $f(z)$ has a pole at $z = a$ and $f(z) = \varphi(z)/(z-a)^n$ as in the Proposition 1.1.3, then the integer $n$ is called the *order of the pole* at $z = a$.

If $f(z)$ has a removable singularity at $z = a$, or is defined and analytic at $z = a$, then we sometimes say that $f(z)$ has a *pole of order* 0 at $z = a$.

**Examples 1.1.5.** Here are typical examples.

- **Removable Singularity** The function
$$f(z) = \frac{\sin z}{z}$$
has a removable singularity at $z = 0$.
- **Pole** Let $a \in \mathbf{C}$ and $n \geq 1$. Then
$$f(z) = \frac{1}{(z-a)^n}$$
has a pole of order $n$ at $z = a$.
- **Essential Singularity** The function $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$.

## 2. The Weierstrass-Casorati Theorem

The behaviour of an analytic function near an essential singularity is given by the following theorem (a more general theorem due to Picard will not be done in the course).

**Theorem 2.1** (Weierstrass-Casorati)**.** *Let $f$ be analytic on $B^* = B(a, r) \setminus \{a\}$ and suppose it has an essential singularity at $a$. Then $f(B^*)$ is dense in $\mathbf{C}$.*

*Proof.* Let $B = B(a, r)$. Suppose $f(B^*)$ is not dense in $\mathbf{C}$. Then we can find an open ball $D = B(w_0, \rho)$ such that $D \cap f(B^*) = \emptyset$. It follows that $|f(z) - w_0| > \rho$ for every $z \in B^*$. Thus the function $g$ on $B^*$ given by

$$g(z) = \frac{1}{f(z) - w_0}$$

is bounded for $|g(z)| < 1/\rho$. By Riemann's removable singularity theorem, $g(z)$ is holomorphic on all of $B$. It follows that $f(z) - w_0$ has at worst a pole at $z = a$ contradicting the fact that $f(z)$ has an essential singularity at $z = a$. $\square$

### 3. The Argument Principle

The main result is the following

**Theorem 3.1.** *Let $B = B(a, r)$ and $f$ a non-zero analytic function on the closed disc $\overline{B} = \overline{B}(a, r)$. Suppose no zero of $f$ lies on $C = \partial B$. Let $S = \{a_1, \ldots, a_k\}$ be the set of zeros of $f$ in $B$ and $n_i$ the order of the zero of $f(z)$ at $a_i$ for $i = 1, \ldots k$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{k} n_i$$

*Remark:* The numbers of zeros of $f(z)$ in $B$ is necessarily finite. Indeed if $Z$ is the set of zeroes of $f$ in $\overline{B}$, then $Z$ is finite since it is a discrete subspace of a compact space. Since $S = Z \cap B$, it too must be finite. In fact our hypothesis that $Z \cap C = \emptyset$ further ensures that $S = Z$. Therefore there was no loss of generality in assuming that the set of points in $S$ can be listed as $a_1, \ldots, a_k$.

*Proof.* We can write

$$f(z) = g(z) \prod_{i=1}^{k} (z - a_i)^{n_i}$$

where $g(z)$ analytic on $\overline{B}$ and $g(a_i) \neq 0$ for $i = 1, \ldots, k$. This means $g(z)$ is nowhere vanishing on $\overline{B}$. It follows that the function $g'(z)/g(z)$ is defined and holomorphic on $\overline{B}$.

Now an easy computation shows that

$$(*) \qquad \frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{i=1}^{k} \frac{n_i}{z - a_i}.$$

Applying the operator $\frac{1}{2\pi i} \int_\gamma (-) dz$ to both sides of $(*)$ and noting that Cauchy's theorem for the holomorphic function $g'/g$ gives

$$\int_\gamma \frac{g'(z)}{g(z)} dz = 0,$$

we get the result. We have used the fact that

$$\frac{1}{2\pi i} \int_C \frac{1}{(z - b)} dz = 1$$

for every point $b \in B$. $\square$