LECTURE 7

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As usual, this is only a summary. Not all proofs given in class are here.

1. Winding Number

The *winding number* of a closed path is strictly speaking a topological invariant. In fact many of the proofs we give of statements using winding numbers can be given without the full notion of a winding number of an arbitrary closed path. That said, it is historical, and makes many statements and proofs a little more transparent (at least to function theorists) and requires very little to develop. In fact the only theorem we need is the following.

Theorem 1.1. *Let $\gamma\colon [\alpha,\, \beta] \to \mathbf{C}$ be a closed path and $a \in \mathbf{C} \smallsetminus \gamma^*$. Then*

$$\int_\gamma \frac{dz}{z-a} = 2\pi i n$$

for some $n \in \mathbb{Z}$.

Proof. Write $z(t) = \gamma(t)$ for $\alpha \le t \le \beta$. Define $h\colon [\alpha,\, \beta] \to \mathbf{C}$ by

$$h(t) = \int_\alpha^t \frac{z'(s)ds}{z(s)-a} \qquad (\alpha \le t \le \beta).$$

It is easy to check that

$$\frac{d}{dt}\big[e^{-h(t)}(z(t)-a)\big] = 0 \qquad (\alpha \le t \le \beta).$$

It follows that $e^{-h(t)}(z(t)-a)$ is a constant. Since $z(\alpha) = z(\beta)$, γ being a closed path, it follows that

$$e^{h(\alpha)} = e^{h(\beta)}.$$

Now $h(\alpha) = 0$. It follows that $e^{h(\beta)} = 1$. In other words

$$h(\beta) = 2\pi i n$$

for some integer n. □

Definition 1.1.1. The winding number of a closed path γ with respect to a point a not lying on γ^* is the integer

$$\eta(\gamma,\, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}.$$

1.2. **Winding number on components.** Recall that the *connected components defined by* γ are by definition the connected components of $\mathbf{C} \setminus \gamma^*$. There is, as is easy to verify, only one connected component which is unbounded. More precisely, if S is the Riemann sphere, there is only one connected component of $S \setminus \gamma^*$ containing ∞, and the intersection of this open set with $\mathbf{C}$ is the unique unbounded connected component defined by γ.

Proposition 1.2.1. *Let γ be a closed path in $\mathbf{C}$. Then the winding number of γ is constant on the connected components defined by γ and is zero on the unbounded component.*

Proof. The map $a \mapsto (1/2\pi i) \int_\gamma dz/(z-a)$ is continuous (in fact analytic) on $\mathbf{C} \smallsetminus \gamma^*$. Being integer valued it is locally constant. Thus it is constant on the connected components defined by γ^*. To show it is zero on the unbounded component U of $\mathbf{C} \smallsetminus \gamma^*$, we pick a positive real number ϵ. Since U is unbounded, and since γ^* is compact, there exists $a \in U$ such that

$$\min_{z \in \gamma^*} |z-a| > \frac{1}{\epsilon}.$$

Let L be the length of γ, i.e., $L = \int_\gamma |dz|$. Then

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \right| \leq \frac{1}{2\pi} L\epsilon.$$

It follows that $\eta(\gamma, a) = 0$. □

Remark 1.2.2. Let $a \in \mathbf{C}$ and $z_0 \in \mathbf{C} \smallsetminus \{a\}$. Recall that we have a canoncial isomorphism

$$(*) \qquad \pi_1(\mathbf{C} \smallsetminus \{a\}, z_0) \xrightarrow{\sim} \mathbb{Z}$$

where the pre-image of $1 \in \mathbb{Z}$ is the circle C with centre a, starting and ending at z_0 with the orientation of C being the counter-clockwise direction. We will see later in the course that $(*)$ is the map

$$[\gamma] \mapsto \eta(\gamma, a)$$

where γ is a closed path representing the class $[\gamma]$. We point out that there is always a closed path (i.e., a C^1 closed curve starting and ending at z_0) which represents the class $[\gamma]$. That two such paths representing $[\gamma]$ will yield the same winding number is a form of Cauchy's theorem, and we will do this later.

2. **Local properties of analytic functions**

Fix a region Ω throughout the discussion in this section. Local properties of an analytic function depend on the following Lemma.

2.1. **Orders of vanishing.**

Lemma 2.1.1. *Let $a \in \Omega$ and suppose $f(z)$ is a non-zero analytic function on Ω. Then there exists a unique analytic function $g(z)$ on Ω such that*

(i) $g(a) \neq 0$,
(ii) $f(z) = (z-a)^n g(z)$ *for* $z \in \Omega$ *for some non-negative integer* n.

The non-negative integer n in (ii) is unique.

Proof. Suppose g and h are analytic functions in a neighbourhood of a, neither vanishing at a, and suppose we have non-negative integers l and k such that

$$(z-a)^l g(z) = (z-a)^k h(z)$$

in a neighbourhood of a. Then it is easy to see that $l = k$. Indeed, without loss generality suppose $l \geq k$. Then in a neighbourhood of a, $(z-a)^{l-k} g(z) = h(z)$ and since $h(a) \neq 0$ we have $l = k$. The uniqueness assertion about the integer n in (ii) follows immediately. The uniqueness of g, if it exists, is also then clear, for $g(z)$ is necessarily then the unique continuous extension of the holomorphic function $f(z)/(z-a)^n$ on $\Omega \setminus \{a\}$. It remains to show that there exists a g satisfying (i) and (ii). It is enough to show this in a ball B centred at a contained in Ω, for if g exists on B it agrees with $f(z)/(z-a)^n$ on $B \setminus \{a\}$ and hence extends holomorphically to Ω. On B we have a power series expansion of $f(z)$, say

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k.$$

Since f is a non-zero function, not all the c_k's are zero. Let n be the smallest index k such that $c_k \neq 0$. Then it is apparent that $f(z) = (z-a)^n g(z)$ where $g(z) = \sum_{k=n}^{\infty} c_k (z-a)^{n-k}$, and clearly the series$\sum_{k=n}^{\infty} c_k (z-a)^{n-k}$ has the same radius of convergence as $\sum_{k=0}^{\infty} c_k (z-a)^k$ has, so that in particular g is defined on B. □

Definition 2.1.2. The integer n in the Lemma is called the *order of f at a*.

2.2. **The Identity Principle.**

Lemma 2.2.1. *Let $f(z)$ be analytic on Ω and suppose there exists a point a in Ω such that $f^{(n)}(a) = 0$ for $n \geq 1$. Then f is a constant.*

Proof. Let

$$S = \{z \in \Omega \mid f^{(n)}(z) = 0 \text{ for } n \geq 1\}.$$

Then S is non-empty (for $a \in S$), and clearly closed, being an intersection of closed sets. On the other hand, if $b \in S$, then the power series expansion of $f(z)$ around b is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(z-b)^n$$

and this has only one non-zero coefficient, namely the constant coefficent. It follows that f is a constant in an open disc D around b, in particular $D \subset S$. It follows that S is open. Since Ω is connected, this means $S = \Omega$, i.e., $f^{(n)}(z) = 0$ for all $n \geq 1$ and all $z \in \Omega$. In particular $f'(z) = 0$ for every $z \in \Omega$, whence f is a constant. □

Theorem 2.2.2 (The Identity Principle)**.** *Let $a \in \Omega$ and let $\{z_n\}$ be a sequence in Ω such that $z_n \to a$ as $n \to \infty$ with $z_n \neq a$ for an infinite number of n. If f is analytic on Ω and $f(z_n) = 0$ for every n, then $f \equiv 0$ on Ω.*

Proof. By hypothesis have a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \neq a$ for any k. Replacing $\{z_n\}$ by the subsequence $\{z_{n_k}\}$ if necessary, we may assume $z_n \neq a$ for any n.

If f is not identically zero, then by Lemma 2.1.1 we can write $f(z) = (z-a)^n g(z)$ with n a non-negative integer and g is a holomorphic function on Ω such that $g(a) \neq 0$. Thus

$$0 = f(z_n) = (z_n - a) g(z_n) \qquad (n \in \mathbb{N}).$$

It follows that $g(z_n) = 0$ for every n, whence $g(a) = 0$, giving a contradiction. $\square$

Example 2.2.3. Recall we defined the exponential function e^z last time. It is an entire function. Define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The functions $\cos z$ and $\sin z$, from their definitions, are entire. Moreover, their restrictions to the real axis are the usual trignometric functions, i.e., cosine and sine from what we proved last time. We claim that

$$\cos^z + \sin^z = 1$$

for every $z \in \mathbf{C}$. Consider the entire function $f(z) = \cos^z + \sin^2 z - 1$. It is zero when restricted to $\mathbf{R}$. Therefore by the Identity Principle it is zero everywhere, proving the claim.

This example also helps us understand the term "Identity Principle". It was originally called *the principle of permanence of functional relations* or *the principle of permanence of functional identities*. The idea being that that functional identities (or relations) like $\cos^2 t + \sin^2 t = 1$ on a subset of $\mathbf{C}$ with a limit point are *permanent*, i.e., their validity is on the entire domain where both sided of the identity are defined.