# LECTURE 6

**Date of Lecture:** January 19, 2017

As usual, this is only a summary. Not all proofs given in class are here.

Recall that for a path $\gamma\colon [a,\, b] \to \mathbf{C}$, the symbol $\int_\gamma f(z)|dz|$ is shorthand for the integral of $f\circ\gamma$ with respect to the "arc-length measure", i.e., $\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|dt$. Equivalently, it is the integral of $f\circ\gamma$ with respect to the total variation measure on $[a,b]$ associated to the measure $\mu$ on $[a,\, b]$ (together with the usual Lebesgue sigma-algebra) given by $E \mapsto \int_E \gamma'(t)dt$. Indeed, we have $\int_\gamma f(z)dz = \int_a^b (f\circ\gamma)d\mu$ and the total variation measure $|\mu|$ of $\mu$ has Radon-Nikodym derivative $|\gamma'|$.

## 1. Maximum modulus

**1.1.** Suppose $f(z)$ is analytic on a region $\Omega$ and $|f(z)|$ achieves a maximum at $a\in\Omega$. Say

$$|f(a)| = M. \tag{1.1.1}$$

Let $C$ be a circle of radius $R$ centred at $a$ such that the closed disc $\overline{B}(a,\, R)$ is contained in $\Omega$. We have

$$\begin{aligned} 2\pi RM = 2\pi R|f(a)| &= R\left|\int_C \frac{f(\zeta)d\zeta}{\zeta-a}\right| \\ &\le R\int_C \frac{|f(\zeta)|}{|\zeta-a|}|d\zeta| \\ &\le \int_C |f(\zeta)||d\zeta| \\ &\le \int_C Md|\zeta| \\ &\le 2\pi RM. \end{aligned}$$

Thus all the inequalities above collapse to equalities, yielding,

$$\int_C |f(\zeta)||d\zeta| = \int_C Md|\zeta|. \tag{1.1.2}$$

Now $M-|f|\ge 0$ on $\overline{B}(a,\, R)$ and is moreover continuous. Therefore (1.1.2) shows that

$$|f(\zeta)| = M \qquad (\zeta\in C).$$

Since $C$ was an arbitrary circle centred at $a$ such that the closed disc it bounds is contained in $\Omega$ we have

$$|f(\zeta)| = M \qquad (\zeta\in \overline{B}(a,\, R)). \tag{1.1.3}$$

In particular the set

$$T = \{z\in\Omega \mid\ |f(z)| = M\}$$

is open, since every point in $T$ has a disc around it contained in $T$. On the other hand it is clearly closed, since $|f|$ is continuous on $\Omega$. Since $\Omega$ is a region, this means $T = \Omega$, for $T$ is non-empty (for $a \in T$)). We have therefore proved the following.

($*$) *If $f$ is analytic on a region $\Omega$ and $|f|$ has a maximum in $\Omega$ then $|f|$ is a constant.*

In fact one can prove more.

**Theorem 1.1.4** (The Maximum Modulus Theorem)**.** *If $f$ is analytic on a region $\Omega$ and $|f|$ has a maximum in $\Omega$ then $f$ is a constant.*

*Proof.* Let $f = u + iv$ be the usual decomposition of $f$ into its real and imaginary parts. As usual, let $x$ and $y$ denote the real and imaginary coordinate variables on $\Omega$ and $z$ the complex coordinate variable on $\Omega$. From ($*$) we see that $|f| = K$ a constant. If $K = 0$ there is nothing to prove. Suppose $K > 0$. We have $u^2+v^2 = K^2$ whence

$$uu_x + vv_x = uu_y + vv_y = 0. \tag{1.1.4.1}$$

Now

$$\begin{aligned} \bar{f}f' &= (u - iv)(u_x + iv_x) \\ &= uu_x + vv_x + i(uv_x - vu_x) \\ &= uu_x + vv_x - i(uu_y + vv_y) && \text{(Cauchy-Riemann)} \\ &= 0 && \text{(by (1.1.4.1)).} \end{aligned}$$

Since $|\bar{f}| = K > 0$, therefore $\bar{f}$ is nowhere vanishing on $\Omega$ and we get $f' \equiv 0$ on $\Omega$. This proves the theorem since $\Omega$ is connecred. □

**Corollary 1.1.5.** *If $f$ is non-constant then $|f|$ cannot attain a maximum in $\Omega$. In particular if $\Omega$ is bounded and $f(z)$ is analytic on $\overline{\Omega}$, then $M := \max_{z\in\overline{\Omega}} |f(z)|$ is attained by $|f|$ on the boundary $\overline{\Omega} \smallsetminus \Omega$.*

*Proof.* Obvious. □

**Remark 1.1.6.** We will give a different and a more natural proof later using the open mapping theorem. The theorem and its corollary above are both referred to as the *maximum modulus principle.*

## 2. Schwarz's Lemma

As usual $\Delta$ will denote the open unit disc centred at the origin, and $\bar{\Delta}$ its closure.

**Theorem 2.1** (Schwarz's Lemma)**.** *Suppose $f\colon \Delta \to \Delta$ is analytic with $f(0) = 0$. Then*

$$|f(z)| \leq |z| \qquad (z \in \Delta). \tag{2.1.1}$$

*Moreover, if $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then*

$$f(z) = cz \tag{2.1.2}$$

*with $c$ a constant of absolute value* 1*.*

*Proof.* Define $g\colon \Delta \to \mathbf{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & z \in \Delta^* := \Delta \smallsetminus \{0\} \\ \\ f'(0) & z = 0. \end{cases}$$

The map $g$ is continuous on $\Delta$ and holomorphic on $\Delta^*$. By Riemann's removable singularities theorem, $g$ is holomorphic on $\Delta$. Let $0 < r < 1$. Applying the maximum modulus principle for $g$ on $\overline{B}(0,r)$ we get

$$|g(z)| \leq \max_{\{|\zeta|=r\}} |g(\zeta)| \leq \frac{1}{r} \qquad (z \in \overline{B}(0,r)).$$

Letting $r \to 1$ we get $|g| \leq 1$ on $\Delta$, whence $|f(z)| \leq |z|$ for $z \in \Delta$. If either of the two conditions in the second part of the theorem is satisfied then $|g|$ attains its maximum, namely the value 1, in $\Delta$, whence by the maximum modulus principle $g$ is a constant $c$ with $|c| = 1$. This proves the second part. □

## 3. Exponentials

**3.1. Product of power series.** Suppose $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ are two convergent power series and $\overline{B} = \overline{B}(0,r)$ is in the intersections of the two discs of convegence. Then $fg$ is analytic on $\overline{B}$ and hence has a power series expansion around 0 in $\overline{B}$

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

We know (using the Leibnitz Rule for differentiation in the second line) that

$$\begin{aligned} c_n &= \frac{(fg)^{(n)}(0)}{n!} \\ &= \frac{\sum_{i=0}^{n} \binom{n}{i} f^{(i)}(0) g^{(n-i)}(0)}{n!} \\ &= \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} \frac{g^{(n-i)}(0)}{(n-i)!} \end{aligned}$$

thus giving

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}. \tag{3.1.1}$$

**3.2. Exponentials.** Consider the power series $E(z) = \sum_{n=0}^{\infty} z^n/n!$. Since $(n+1)!/n! = n \to \infty$ as $n \to \infty$, $E(z)$ has radius of convergence $R = \infty$, whence $E(z)$ represents an entire function. Let $\alpha$ and $\beta$ be complex numbers. Applying (3.1.1) to the power series $E(\alpha z)$ and $E(\beta z)$ we see that $E(\alpha z)E(\beta z) = E((\alpha+\beta)z)$ whence, setting $z = 1$, we get

$$E(\alpha)E(\beta) = E(\alpha + \beta) \qquad (\alpha, \beta \in \mathbf{C}). \tag{3.2.1}$$

Clearly

- $E(0) = 1$,
- $E'(z) = E(z)$.

It follows that

$$E(s) = e^s \qquad (s \in \mathbf{R}),$$

the usual exponential, and

$$E(it) = \cos(t) + i\sin(t) \qquad (t \in \mathbf{R}).$$

Indeed, the relationships $E'(s) = E(s)$ and $E(0) = 0$ force the relation $E(s) = e^s$ for $s \in \mathbf{R}$, for $e^s$ is the unique solution of the differential equation

$$\frac{dy}{ds} = y, y(0) = 1.$$

Also, by the chain rule, if $h(t) = E(it)$ for $t \in \mathbf{R}$, then $h'(t) = iE'(it) = iE(it) = ih(t)$. Thus if $h(t) = C(t) + iS(t)$ with $C$ and $S$ real-valued, then $ih(t) = h'(t) = C'(t) + iS'(t)$, giving $C'(t) = -S(t)$ and $S'(t) = C(t)$. Thus $C$ and $S$ are solutions of the

$$\frac{d^2y}{dt^2} = -y$$

and the initial conditions force the relations $C(t) = \cos(t)$ and $S(t) = \sin(t)$. We define the exponential function as

$$e^z := E(z). \tag{3.2.2}$$

In particular, with this definition, we have *Euler's formula*

$$e^{it} = \cos(t) + i\sin(t) \qquad (t \in \mathbf{R}).$$

In general if $z = x + iy$ is the break-up of a complex number $z$ into its real and imaginary parts, then

$$e^z = e^y(\cos(x) + i\sin(x)).$$