LECTURE 5

Date of Lecture: January 18, 2017

As usual, this is only a summary. Not all proofs given in class are here.

1. **This and that**

Here are some facts.

1) If $\sum c_n(z-a)^n$ is a power series with radius of convergence R, and if $l = \lim_{n\to\infty}|a_n/a_{n+1}|$ exists as an extended real number (i.e., ∞ is allowed as a limit) then $R = l$. We will give a proof in a separate note. In the meanwhile feel free to use it in doing problems in HWs, quizzes, and exams.

2) The version of Cauchy's theorem we proved is due to Goursat. Before Goursat, one assumed that a holomorphic function was C^1 in order to prove Cauchy's theorem.

3) We have repeatedly used the fact that if $f_n\colon X \to \mathbf{C}$, $n = 0, 1, \ldots$ is a sequence of functions on a set X and we have a convergent series M_n of non-negative numbers such that

$$|f_n(x)| \le M_n \qquad (x \in X)$$

then $\sum_n f_n$ converges uniformly on X. The proof has been given (a few times) in class. What I omitted to mention is that this is called the *Weierstrass M-test*. The argument is as follows. First, by the dominated convergence theorem for the counting measure on the non-negative integers, $\sum_n f_n < \infty$. Next, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n>N} M_n < \epsilon$. It follows that

$$\left.\begin{aligned} \Big\|\sum_{n=0}^{\infty} f_n - \sum_{n=0}^{m} f_n\Big\|_\infty &\le \Big\|\sum_{n>m} f_n\Big\|_\infty \\ &\le \sum_{n>m}\|f_n\|_\infty \\ &\le \sum_{n>m} M_n \\ &\le \sum_{n>N} M_n \\ &< \epsilon \end{aligned}\right\} \qquad (m \ge N)$$

giving the required uniform convergence.

2. **Two well-known theorems**

2.1. **Liouville's Theorem.** The following is called *Liouville's Theorem.*

Theorem 2.1.1. *An entire bounded function is constant.*

Proof. Let $f(z)$ be entire and bounded, say $|f(z)| \leq M$ for all $z \in \mathbf{C}$ where M is a finite constant. Let $a \in \mathbf{C}$ and let R be any positive number. The Cauchy estimates on the disc $\overline{B}(a, R)$ gives us

$$|f'(a)| \leq \frac{M}{R}.$$

Now let $R \to \infty$ to conclude that $f'(a) = 0$. Thus f is a constant. $\square$

Remark 2.1.2. An alternate proof is obtained by noting that $|f^{(n)}(0)| \leq n!M/R^n$ for all $n \geq 0$. If $n \geq 1$, one sees that $n!M/R^n \to 0$ as $n \to \infty$, whence $f^{(n)}(0) = 0$ for $n \geq 1$. The usual Taylor's series expansion around 0, which is valid on all of $\mathbf{C}$ since f is entire, gives us the result.

2.2. **Riemann's Removable Singularities Theorem.** The following is usually attributed to Riemann.

Theorem 2.2.1. *Suppose Ω is a region, $a \in \Omega$ a point and $f\colon \Omega \smallsetminus \{a\} \to \mathbf{C}$ an analytic function such that $(z-a)f(z) \to 0$ as $z \to a$. Then f can be extended to an analytic function on all of Ω.*

Proof. Let B be an open disc containing a such that $\overline{B} \subset \Omega$. Let $C = \partial\overline{B}$ be the bounding circle oriented in the usual way. From Problem 5 of HW 1, or equivalently from Theorem 1.1 of Lecture 2, we see that

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta \tag{2.2.1.1}$$

defines an analytic function on B. We claim that g extends f. In fact Cauchy's integral formula (if applicable) shows that $g(z) = f(z)$ if $z \neq a$. To see Cauchy's integral formula applies we offer the following standard (and by now familiar) argument. Fix $z \in B \smallsetminus \{a\}$ and define $h\colon B \smallsetminus \{a, z\} \to \mathbf{C}$ by

$$h(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \in \{a, z\} \\ \\ f'(z) & \text{if } w = z \end{cases}$$

Then one checks easily that $(w-b)h(w) \to 0$ as $w \to b$ for $b \in \{a, z\}$. Morever h is analytic on $B \smallsetminus \{a, z\} \to \mathbf{C}$. Thus by Cauchy's theorem $\int_C h(\zeta)d\zeta = 0$, whence we see that the right side of (2.2.1.1) yields $f(z)$ for $z \neq a$. $\square$

3. Morera's Theorem

3.1. We have shown that if $f(z)$ is analytic on an open set Ω in $\mathbf{C}$ then $f(z)$ is infinitely differentiable, for $f(z)$ has a power series expansion around every point in Ω. As function of two real variables, such an f is C^∞. It is clear, from the results we have proven so far that we have the following.

Theorem 3.1.1. *Let f be a complex-valued function on a region Ω. The following are equivalent:*

(a) *f is analytic.*
(b) *f is C^1 and analytic.*
(c) *$f = u + iv$, where u, v are real-valued C^1 functions satisfying the Cauchy-Riemann equations.*

Another important theorem is the following:

Theorem 3.1.2 (Morera's Theorem)**.** *Suppose f is a continuous $\mathbf{C}$-valued function on a region Ω such that*

$$\int_\gamma f(z)dz = 0$$

for every closed path γ in Ω. Then f is analytic.

Proof. An f satisfying the hypotheses must have a primitive F on Ω. Since F is analytic, so is its derivative f, for analytic functions are infinitely differentiable. □

Corollary 3.1.3. *Suppose $\{f_n\}$ is a sequence of analytic functions on Ω converging uniformly on compact subsets to a function f. Then f is analytic.*

Proof. If γ: $[a, b] \to \Omega$ is a path and , then as γ^* is compact, $f_n \to f$ uniformly on γ^* as $n \to \infty$, or what is the same thing, $f_n \circ \gamma \to f \circ \gamma$ converges uniformly on $[a, b]$. We therefore have, with $L = \int_a^b |\gamma'|dt$ ($L =$ the length of the γ),

$$\begin{aligned} \left| \int_\gamma (f_n(z) - f(z)dz \right| &\leq \int_a^b \|f_n \circ \gamma - f \circ \gamma\|_\infty |\gamma'(t)|dt \\ &\leq \|f_n \circ \gamma - f \circ \gamma\|_\infty L \\ &\to 0, \end{aligned}$$

as $n \to \infty$. In particular

$$(*) \qquad \int_\gamma f(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz.$$

Let B be an open ball in Ω. From Cauchy's theorem and $(*)$ we conclude that $\int_\gamma f(z)dz = 0$ for every closed path γ in B. By Morera's Theorem it follows that f is analytic on B. Since analyticity is a local property, and such open balls B cover Ω, we are done. □

Remark 3.1.4. Sometimes the term Morera's theorem is used for the statement with the same conclusion as our Morera's theorem but with the weaker hypothesis that $\int_\gamma f(z)dz = 0$ for every triangle γ in Ω. Note that a weaker hypothesis makes for a stronger statement if the conclusion is the same, for you are getting away with less. But in this case, the strengthening is mild. Indeed, the integrals of f over rectangles are zero (every rectangle can be broken up into two triangles via a diagonal), and then a familiar argument shows that locally primitives exist, and that is enough to conclude that f is analytic. We should point out that the converse is also true, namely, if $f(z)$ is analytic then its integral over a triangle in its domain of definition is zero. The proof is the same (dividing triangles rather than rectangles) as Goursat's proof we gave for Cauchy's theorem over rectangles. So often the statement $f(z)$ *is analytic if and only if its integral over every triangle in its domain of definition is zero* is referred to as the *Goursat-Morera theorem.*