LECTURE 4

Date of Lecture: January 12, 2017

As usual, this is only a summary. Not all proofs given in class are here.

1. Cauchy's theorem in a Disc

Theorem 1.1. *Let $B = B(a, r)$ be a disc of radius r centered at $a \in \mathbf{C}$. Let B' be the region obtained from B by deleting a finite subset (possibly empty) S of B. If $f(z)$ is analytic on B' and $\lim_{z\to\zeta}(z-\zeta)f(z) = 0$ for every $\zeta \in S$ then*

$$\int_\gamma f(z)dz = 0$$

for every closed path γ in B'.

Proof. Pick a base point $z_\circ$ in B'. For any $z \in B'$ we can find a path σ_z in B' starting at $z_\circ$, consisting of segments which are either horizontal or vertical line segments, and ending at z. One checks easily that from Cauchy's theorem for a rectangle that

$$F(z) = \int_{\sigma_z} f(w)dw$$

does not depend upon σ_z. The shape of B (that it is a disc) plays a crucial role here. Moreover, we can arrange matters so that the last segment of σ_z is either vertical or horizontal. Picking the last segment to be horizontal, we get by the fundamental theorem of Calculus (for real variables) gives $(\partial F/\partial x)(z) = f(z)$. On the other hand, picking the last segment to be vertical we get (again by the fundamental theorem of calculus) that $(\partial F/\partial y)(z) = if(z)$. Thus $(\partial F/\partial x)(z) = -i(\partial F/\partial x)(z)$. This means F is analytic and $F'(z) = (\partial F/\partial x)(z) = f(z)$. The details are similar to those given in the proof of Proposition 3.1.1 in Lecture 2. $\square$

2. Cauchy's Integral Formula and Power Series

Let r be a positive real number, a a point in $\mathbf{C}$, and let $B = B(a, r)$, $\overline{B}$, the closure of B in $\mathbf{C}$, i.e., the closed disc of radius r around a, and let C be the circle $\{|z - a| = a\}$. As always, $\int_C$ will mean integrating along the positive (counter-clockwise) direction of C.

2.1. **Basic Computation.** The following formula is basic to complex analysis:

$$\int_C \frac{dz}{z-a} = 2\pi i \tag{2.1.1}$$

.

For the proof, note that without loss of generality we may assume $a = 0$. Now use the parameterisation $\theta \mapsto z(\theta) = r(\cos\theta + i\sin\theta)$ of the bounding circle C, as the parameter θ varies over $[0, 2\pi]$. One checks easily that $z'(\theta) = iz(\theta)$ whence $\int_0^{2\pi} z'(\theta)d\theta/z(\theta) = i\int_0^{2\pi} d\theta = 2\pi i$.

From Theorem 1.1 of Lecture 2, or from Problem 5 of HW-1, we see that

$$g(w) = \int_C \frac{dz}{z-w}$$

is an analytic function on $U = \mathbf{C} \smallsetminus C$, and the derivative of g on U is

$$g'(w) = -\int_C \frac{dw}{(z-w)^2}.$$

Now, for fixed z, the expression $(z-w)^{-2}$ regarded as a function of w has a primitive on U. Indeed

$$\frac{1}{(z-w)^2} = \frac{d}{dw}\left[\frac{1}{z-w}\right].$$

Thus $g' = 0$ on U. It follows that g is constant on the connected components determined by C (i.e., the connected components of $\mathbf{C} \smallsetminus C$). There are two components, one of which is B. Now by (2.1.1), $g(a) = 2\pi i$. Hence we get

$$\int_C \frac{dz}{z-w} = 2\pi i \qquad (w \in B) \tag{2.1.2}$$

2.2. **The integral formula for a disc.** The following theorem is critical for deducing the local properties of analytic functions.

Theorem 2.2.1. *Let f be analytic on closed disc $\overline{B}$. Then*

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - w}d\zeta \qquad (w \in B).$$

Proof. Fix $w \in B$. For $z \neq w$ in the domain of f, define $g(z)$ be the formula

$$g(z) = \frac{f(z) - f(w)}{z-w}.$$

Then $g(z)$ is analytic in $\overline{B} \smallsetminus \{w\}$. Moreover, $\lim_{z\to w}(z-w)g(z) = 0$. Thus Cauchy's theorem on the disc applies to g and we have $\int_C g(\zeta)d\zeta = 0$. It follows that

$$\int_C \frac{f(\zeta)d\zeta}{\zeta - w} = \int_C \frac{f(w)d\zeta}{\zeta - w} = f(w)(2\pi i)$$

where we have used (2.1.2) for the last equality. □

2.3. **Power Series.** Let $e(\theta)$ be the map $\theta \mapsto \cos\theta + i\sin\theta$, $0 \le \theta \le 2\pi$. Then $\varphi := a + re$ is a paramaterisation of C which respects its orientation. Now suppose, as in Theorem 2.2.1, f is analytic on $\overline{B}$. The map f along with the parameterisation φ induces a complex measure μ on $[0,\, 2\pi]$ given by $E \mapsto \int_E (f\circ\varphi)\varphi' dm$ where m is the Lebesgue measure and E a Lebesgue measurable set. By Theorem 1.1 of Lecture 2 (with $X = [0,\, 2\pi]$, $\mathscr{F}$ the Lebesgue σ-algebra, Ω the domain of $f(z)$, and φ and μ as defined above) we see that if $w \in B$ then f has a power-series expansion in any disc contained in B centred at w:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-w)^n$$

where

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - w)^{n+1}}, \qquad (n \ge 0).$$

Moreover by Corollary 2.1.2 of Lecture 1 we have $c_n = f^{(n)}(w)/n!$. Thus we get the following power series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^n \tag{2.3.1}$$

for every w in the domain of f. The expansion is valid in any disc centred at w contained in the domain of f as a little thought shows. Note that the formula for c_n above gives us

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - w)^{n+1}}, \qquad (n \geq 0) \tag{2.3.2}$$

a formula that can also be deduced from problem 5 of HW-1.

2.4. **Cauchy estimates.** The following gives an estimate for growth of derivatives.

Proposition 2.4.1. *Suppose f is analytic on $\overline{B}$ and $\sup_{z\in C}|f(z)| \leq M < \infty$. Then*

$$|f^{(n)}(a)| \leq \frac{n!M}{r^n} \qquad (n \geq 0).$$

Proof. We have by (2.3.2)

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$$

for each $n \geq 0$. It follows that for each $n \geq 0$ we have

$$\begin{aligned} |f^{(n)}(a)| &\leq \frac{n!}{2\pi}\int_0^{2pi} \frac{M|re(\theta)|}{|re(\theta)|^{n+1}}d\theta \\ &= \frac{n!}{2\pi}\frac{Mr}{r^{n+1}}2\pi r \\ &= \frac{n!M}{r^n}. \end{aligned}$$

□

References

[A] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New-York, 1979.