LECTURE 3

Date of Lecture: January 11, 2017

As usual, this is only a summary. Not all proofs given in class are here.

1. **Primitives**

The following two conditions on a continuous function f on a region Ω are trivially equivalent:

- For a [ath $\gamma\colon [a,\, b] \to \Omega$ in Ω, the integral $\int_\gamma f(z)dz$ depends only on the end-points $\gamma(a)$ and $\gamma(b)$ of γ and not on γ.
- The integral $\int_\gamma f(z)dz = 0$ for every closed path γ in Ω.

If f satisfies the above we say f has *path-independent* integrals, or sometimes, the differential $f(z)dz$ has path independent integrals.

Last lecture we proved that if f as above has path-independent integrals then f has a primitive on Ω. The converse is also true. Here is the complete statement

Theorem 1.1. *Let Ω be a region in $\mathbf{C}$. A complex-valued continuous function on Ω has primitive in Ω if and only if f has path-independent integrals.*

Proof. We have already proven the "if" part in the last lecture. We now prove the "only if part". Suppose f is continuous on Ω and has a primitive F in Ω. Then F satisfies the Cauchy-Riemann equations and hence

$$f = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}.$$

It follows that

$$\begin{aligned} f(z)dz &= \frac{\partial F}{\partial x}(dx + idy) \\ &= \frac{\partial F}{\partial x}dx + i\frac{\partial F}{\partial x}dy \\ &= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \qquad \text{(Cauchy-Riemann)} \\ &= dF. \end{aligned}$$

It follows that if $\gamma\colon [a,\, b] \to \Omega$ is a path, with $p = \gamma(a)$ and $q = \gamma(b)$, then

$$\begin{aligned} \int_\gamma f(z)dz = \int_a^b \gamma^*(f(z)dz) &= \int_a^b \gamma^*(dF) \\ &= F(\gamma(b)) - F(\gamma(a)) \\ &= F(q) - F(p) \end{aligned}$$

as required. For a differential ω on Ω, the differential $\gamma^*\omega$ makes sense only on $[a,\, b] \smallsetminus \{t_0, \ldots, t_n\}$ where the t_i's are the points of partition of $[a,\, b]$ such that on $[t_{i-1},\, t_i]$ the map γ is differentiable with continuous derivative. Be that as it may, one can still apply Stokes' Theorem to $\gamma_i^*\omega$ ($\gamma_i = \gamma|_{(t_{i-1},t_i)}$) and if these differentials extend to the manifold with boundary $[t_{i-1}, t_i]$ then we can add to make sense of $\int_a^b \gamma^*\omega$. In our case these hypotheses are satisfied.

Another way is to use the parameterisation $t \mapsto \gamma(t) = x(t) + iy(t)$, where $x(t)$ and $y(t)$ are the real and imaginary parts of $\gamma(t)$. By the chain rule for the composition $g = F \circ \gamma$, we have (on each subinterval $[t_{i-1}, t_i]$)

$$\begin{aligned} g'(t) &= F_x(\gamma(t))x'(t) + F_y(\gamma(t))y'(t) \\ &= f(\gamma(t))x'(t) + if(\gamma(t))y'(t) \\ &= f(\gamma(t))\gamma'(t). \end{aligned}$$

It follows that $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b g'(t)dt = g(b) - g(a)$ giving the result. □

2. **Cauchy's Theorem**

2.1. **Cauchy's theorem for a rectangle.** Let R be a closed bounded rectangle in $\mathbf{C}$ with verticles $a+ic$, $b+ic$, $a+id$, $b+id$, with $a < b$ and $c < d$. Let R° be the interior of R an ∂R the boundary of R oriented in the usual way (with R° falling to the left when one travels along the path, which we do at speed $s = 1$).

Theorem 2.1.1. *Let f be analytic on R. Then*

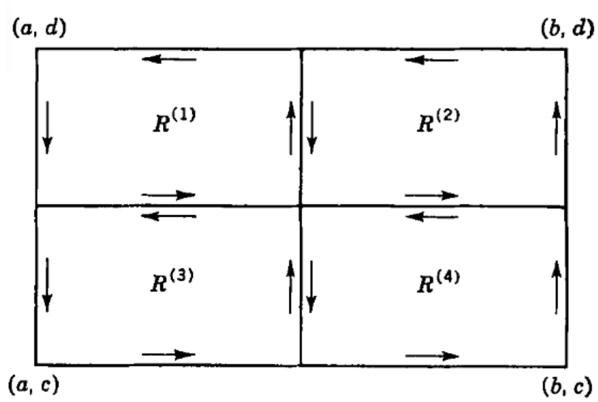
$$\int_{\partial R} f(z)dz = 0.$$

The result continues to hold even if f is only defined and analytic on $\Omega \setminus S$ where $S = \{\zeta_1, \ldots, \zeta_m\}$ is a finite subset of R°, provided $(z - \zeta_i)f(z) \to 0$ as $z \to \zeta_i$ for $i = 1, \ldots, m$.

Proof. Let us first prove the theorem when f is analytic on R, i.e., when $S = \emptyset$. For any sub-rectangle of Q of R (with $Q = R$ a possibility), with Q closed and sides parallel to the real and imaginary axes, define

$$\eta(Q) = \int_{\partial Q} f(z)dz.$$

Divide R into four sub-rectangles $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, and $R^{(4)}$ as follows (the picture is the one in [A, Fig.4-2, p.110]):



If an edge is shared between two sub-rectangles then clearly the orientation of the edge induced by one sub-recatngle is opposite to the orientation on the edge induced by the other sub-rectangle. Hence

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})$$

whence there is at least one $R^{(i)}$, call it R_1, such that

$$|\eta(R_1)| \geq \frac{1}{4}|\eta(R)|.$$

One can repeat the process on R_1, i.e., subdivide R_1 into four sub-rectangles in the manner the sub-division was done for R etc., to arrive at a sub-rectangle R_2 such that $|\eta(R_2)| \geq (1/4)|\eta(R_1)|$. Continuing the process we get a family of sub-rectangles $\{R_n\}$, with $R_{n+1} \subset R_n$ such that

$$4^{-n}|\eta(R)| \leq |\eta(R_n)| \qquad (n \geq 1). \tag{2.1.1.1}$$

Let d be the length of a diagonal of R and L its perimeter, and let d_n and L_n be the corresponding quantities for R_n. We then have

$$d_n = 2^{-n}d \quad \text{and} \quad L_n = 2^{-n}L.$$

Since R is a complete metric space and $\{R_n\}$ is a nested family of closed subsets such that $\operatorname{diam}(R_n) = d_n = 2^{-n}d \longrightarrow 0$ and $n \longrightarrow \infty$, we have, by the nested sets theorem there is unique point z^* in the intersection of the R_n, i.e.,

$$\bigcap_{n=1}^{\infty} R_n = \{z^*\}.$$

Now suppose $\epsilon > 0$ is given. There exists a $\delta > 0$ such that $B(z^*, \delta)$ is contained in the domain of f and

$$\left|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\right| < \epsilon \qquad (0 < |z - z^*| < \delta).$$

Thus

$$\Big|f(z) - f(z^*) - f'(z^*)(z - z^*)\Big| < \epsilon|z - z^*| \qquad (|z - z^*| < \delta). \tag{2.1.1.2}$$

We can find $N \geq 1$ such that $L_n = 2^{-n}L \leq \delta/2$ for $n \geq N$. It follows that $R_n \subset B(z^*, \delta)$ for $n \geq N$. Moroever the constant $f(z^*)$ and the analytic function $f'(z^*)(z - z^*)$ have primitives $f(z^*)z$ and $f'(z^*)(z - z^*)^2/2$ and hence

$$\int_{\partial R_n} f(z)dz = \int_{\partial R_n} \Big(f(z) - f(z^*) - f'(z^*)(z - z^*)\Big).$$

For $n \geq N$ it then follows from (2.1.1.2) that

$$\begin{aligned} |\eta(R_n)| = \Big|\int_{\partial R_n} f(z)dz\Big| &< \epsilon \int_{\partial R_n} |z - z^*||dz| \\ &\leq \epsilon d_n \int_{\partial R_n} |dz| \\ &= \epsilon d_n L_n \\ &= \epsilon 4^{-n} dL \end{aligned} \tag{2.1.1.3}$$

From (2.1.1.1) and (2.1.1.3) we conclude that $|\eta(R)| < \epsilon dL$, and since ϵ was an arbitrary positive number, $\eta(R) = 0$.

Now suppose $S \neq \emptyset$. We can find a finite system of lines parallel to the axes which divide R into sub-rectangles Q_j (j varying over some finite index set) such that no ζ_i lies in the boundary of any Q_j and each Q_j contains at most one ζ_i in its interior. For $1 \leq i \leq m$ let $Q^{(i)}$ be the unique sub-rectangle containing ζ_i in its interior. Then it is clear from what we proved above that $\eta(R) = \sum_{i=1}^{m} \eta(Q^{(i)})$, for $\eta(Q_j) = 0$ for any Q_j not containing any ζ_i. Thus without loss of generality we may assume $m = 1$ and set $\zeta = \zeta_1$.

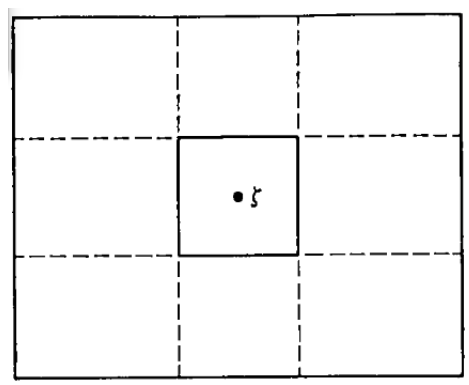
Suppose $\epsilon > 0$ is given. We can find $\delta > 0$ such that $B(\zeta, \delta) \subset R$ and such that $|(z - \zeta)f(z)| < \epsilon$ whenever $0 < |z - z^*| < \delta$. Let Q be a square with side ℓ centred at ζ such that $\ell < \delta/\sqrt{2}$. Then for $z \in \partial Q$, $|z - \zeta| \leq \sqrt{2}\ell/2 < \delta/2$. It follows that $Q \subset B(zeta, \delta)$. Moroever,

$$|z - \zeta| \geq \frac{\ell}{2} \qquad (z \in \partial Q).$$

giving

$$|f(z)| < \frac{\epsilon}{|z - \zeta|} \leq 2\frac{\epsilon}{\ell} \qquad (z \in \partial Q). \tag{2.1.1.4}$$

Subdivide R as follows into nine rectangles, with Q the rectangle containing ζ (picture taken from [A, Fig.4-3, p.112]).



Clearly, from what we proved earlier for functions analytic on closed rectangles, $\eta(P) = 0$ for any sub-rectangle $P \neq Q$ in the above decomposition. It follows that

$$\eta(R) = \eta(Q).$$

From (2.1.1.4) we get

$$|\eta(Q)| < \int_{\partial Q} 2\frac{\epsilon}{\ell}|dz| = 2\epsilon\frac{4\ell}{\ell} = 8\epsilon.$$

Thus $|\eta(R)| = |\eta(Q)| < 8\epsilon$, whence $\eta(R) = 0$. □

Remark 2.1.2. This proof is due to the French mathematician Édouard Goursat, and it appeared in his book on analysis in the first decade of the 20th century. Before Goursat, it was standard to assume that an analytic function is C^1 in addition to being differentiable. This proof shows (as we will see in Lecture 4) that the extra assumption is un-necessary, for an analytic function (as we have defined it) is infinitely differentiable and hence certainly C^1 (in fact C^∞).

References

[A] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New-York, 1979.