LECTURE 26

Date of Lecture: April 17, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

The unit circle will be denoted $\mathbf{T}$ instead of C. As usual Δ will denote the open unit disc. The Riemann sphere will be denoted $\mathbb{P}^1$.

1. Linear Fractional Transformations (continued)

We will use the abbreviation LFT for a linear fractional transformation.

1.1. **Circles and Lines.** First note that the group of LFTs is generated by the three kinds of elements:

(1) Dilatations or transformations of the type
$$T(z) = kz$$
where k is a non-zero constant. Note that $T = T_A$ where $A = \left(\begin{smallmatrix} k & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(2) Translations, i.e., transformations of the type
$$T(z) = z + a \qquad (a \in \mathbf{C})$$
where a is a constant. Here $T = T_A$ with $A = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$.

(3) The reciprocal map
$$T(z) = \frac{1}{z}.$$
In this case $T = T_A$ with $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Indeed, if $T(z) = \dfrac{az+b}{cz+d}$ with $ad - bc \neq 0$, then one reasons as follows. If $c = 0$, then $ad \neq 0$, in particular $d \neq 0$. Dividing the numerator and denominator by d, we may assume $d = 1$. Then $T(z) = az + b$ which is a composite of the type $(2)\circ(1)$. Similarly if $a = 0$, then $bc \neq 0$ so that we may divide the numerator and denominator by b if necessary and assume $b = 1$. This means $T(z) = \dfrac{1}{cz+d}$, which is a composite the type $(3)\circ(2)\circ(1)$. Finally, if neither a nor c is zero, we may divide the numerator and denominator by c if necessary and assume $c = 1$. In this case $T(z) = \dfrac{az+b}{z+d} = \dfrac{b-ad}{z+d} + a$. This is a transform of the type $(2)\circ(1)\circ(3)\circ(2)$.

Proposition 1.1.1. *An LFT transforms a line into either a line or a circle, and a circle to either a line or a circle.*

Proof. Let T be an LFT. If it is of type (1) or (2), the proposition is clearly true for T. We therefore only need to check the proposition for
$$T(z) = \frac{1}{z}.$$
Note that $T(e^{i\theta}z) = e^{-i\theta}T(z)$ for $\theta \in \mathbf{R}$. In particular, if L is a line, and $L' = e^{i\theta}L)$, then $T(L) = e^{i\theta}T(L')$. This means to prove that $T(L)$ is a circle or a line it is enough

to prove that $T(L')$ is a circle or a line. We may therefore assume that L is parallel to the y-axis, say $L = \{a + it \mid t \in \mathbf{R}\}$ for some real constant a. The discussion is simplified if we allow L to have the point ∞ on it, by replacing L by $L \cup \{\infty\}$, and allowing $t = \infty$ so that the parameter t varies in $\mathbf{R} \cup \{\infty\}$.

Then $T(L)$ is has the parameterisation

$$t \mapsto \frac{1}{a+it} = \frac{a}{a^2+t^2} - i\frac{t}{a^2+t^2} \qquad (t \in \mathbf{R} \cup \{\infty\}.$$

If $a = 0$ the above shows that $T(L)$ is clearly the extended imaginary axis (i.e., the extended line $i\mathbf{R} \cup \{\infty\}$). Suppose $a \neq 0$. Make the substitution

$$t = a \tan\theta \qquad (-\frac{\pi}{2} < \theta \leq \frac{\pi}{2})$$

where $t = \infty$ when $\theta = \pi/2$. Then we have

$$\begin{aligned} T(a+it) &= \frac{1}{a}\cos^2\theta + i\frac{1}{a}\tan\theta\cos^2\theta \\ &= \frac{1}{a}\cos^2\theta + i\frac{1}{a}\sin\theta\cos\theta \\ &= \frac{\cos 2\theta + 1}{2a} + i\frac{\sin 2\theta}{2a} \\ &= \frac{1}{2a} + \frac{\cos 2\theta}{2a} + i\frac{\sin 2\theta}{2a}. \end{aligned}$$

Thus $T(L)$ is a circle of radius $1/2a$ centred at $1/2a$. (Note, $T(L)$ passes through 0. Set $\theta = \pi/2$ to see this, if you do not wish to see it geometrically.)

Now suppose C is a circle, say given by

$$(\dagger) \qquad |z - a| = r$$

where $r > 0$ is the radius. Then the equation for $T(C)$ is

$$\left|\frac{1}{w} - a\right| = r$$

where $w = T(z)$. Simplifying, we get $|1 - aw| = r|w|$. Squaring we get

$$(*). \qquad |1 - aw|^2 = r^2|w|^2$$

Set $w = x + iy$ (x, y in $\mathbf{R} \cup \infty$) and let $a = \alpha + i\beta$ be the decomposition of a into its real and imaginary parts. Simplifying $(*)$ we get

$$(**). \qquad (r^2 - |a|^2)x^2 + (r^2 - |a|^2)y^2 + 2\alpha x - 2\beta y - 1 = 0$$

Since the coefficients of x^2 and y^2 are equal, and there are no cross terms (i.e., terms of the kind cxy, c a non-zero constant), therefore $(**)$ is the equation of a circle, unless $r = |a|$, in which $(**)$ represents the line $2\alpha x - 2\beta y = 1$. This completes the proof. Note however that when $|a| = r$, then the original circle C, given by $(\dagger)$, must pass through 0. And that is the real reason why $T(C)$ is a line rather than a circle, for $T(0) = \infty$. □

2. **Reflections revisited**

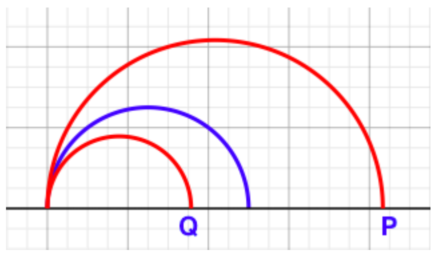
2.1. **Formula for inversion in a circle.** Let C be a circle in $\mathbf{C}$. The reflection R_C with respect to C is called *inversion with respect to C*. Suppose C has radius

ρ and centre $a = \alpha + i\beta$, where α and β are real. The formula is as follows (for x and y real numbers, $z = x + iy$, $z \neq a$)

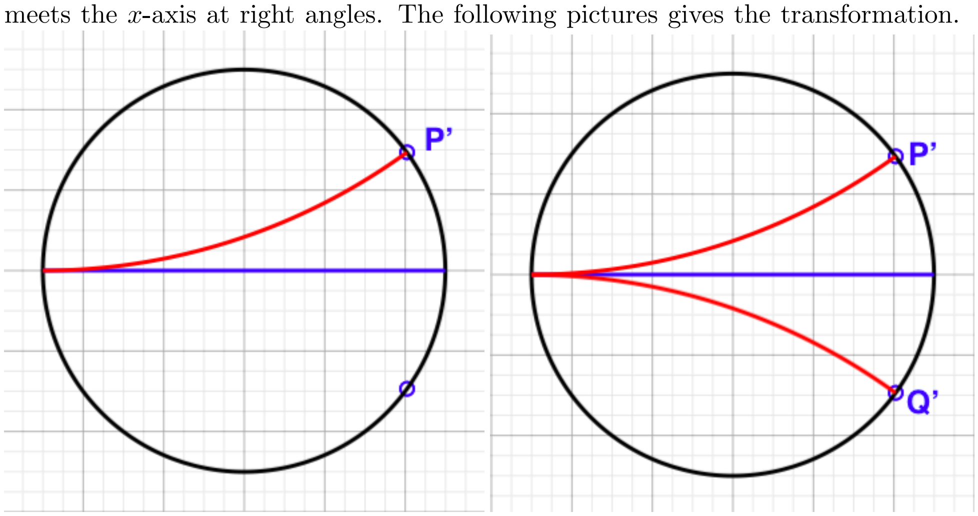
$$(2.1.1) \qquad R_C(z) = a + r^2\left[\frac{x-\alpha}{(x-\alpha)^2 + (y-\beta)^2} + i\frac{y-\beta}{(x-\alpha)^2 + (y-\beta)^2}\right].$$

For $z = a$ we can use the convention that $R_C(a) = \infty$, and to that one can add the convention that $R_C(\infty) = a$. With this we have $R_C \colon \mathbb{P}^1 \to \mathbb{P}^1$. A word of caution — R_C is orientation reversing.

2.2. **Reflection of a line in a circle it meets at right angles.** We now address a question which crops up in the proof of Picard's theorem which we gave last lecture. Consider the picture below with all the curves semi-circles whose diameters are segments of the x-axis.



We would like to reflect the outer red semi-circle with respect to the circle represented by the blue semi-circle. The black line is the x-axis. Our claim is that the reflection Q of P lies on the x-axis as seen in the picture. (In fact the reflection of the outer red semi-circle is the inner red semi-circle.) To prove the claim, use an LFT that transforms the blue circle into the x-axis. Now the blue semi-circle meets the x-axis at both ends at right angles. Therefore so does its transformation. This means the original x-axis (the black line) gets transformed to a circle which meets the x-axis at right angles. The following pictures gives the transformation.

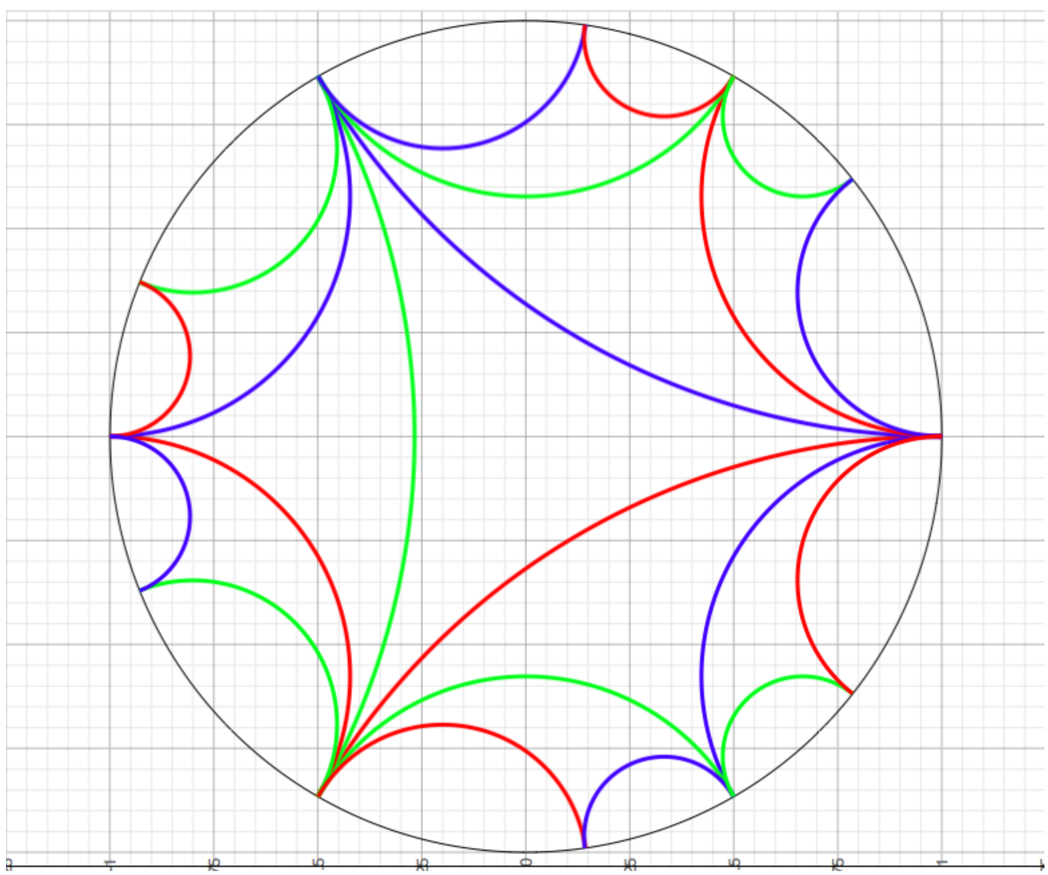


The point P' is the transformation of P, and the colour codes give the rest. The point is this, since the blue line meets the black circle at right angles, it must be a diameter. The lower half of the black circle is the transformation of the diameter along the x-axis of the original blue circle. It follows that the reflection of P', call it Q', must lie on the black circle. Since Q' is he transformation of Q, it follows that Q, i.e., the reflection of P, lies on the x-axis. This proves the claim. What we have really proved is that the reflection of any point on the extended x-axis with respect to the circle represented by the blue semi-circle is again a point on the extended x-axis.

Remark 2.2.1. These conclusions can also be reached by using (2.1.1).

3. Little Picard again

3.1. Here is another version of the proof given last time of Picard's theorem. The picture to contemplate is:



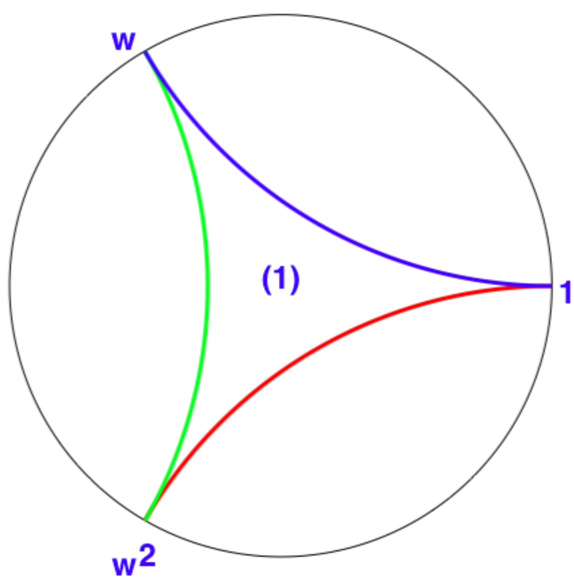
The outer circle (in black) is the unit circle. The inner triangle (the one containing the origin) is such that its arcs meet the unit circle at right angles at the cube roots of unity. unity $1, \omega, \omega^2$, where we use the convention that $\mathrm{Im}(\omega) > 0$. There are two kinds of triangles. Those for which the order blue–green–red defines a counter-clockwise direction, and those for which it defines a clockwise direction. Of the former type there are seven, namely, the original inner triangle and six smaller ones. Of the latter type there are three, namely the three reflections of the original triangles about its bordering arcs. The process can of course be continued indefinitely. All triangles with the blue–green–red order counter-clockwise are even reflections of the original triangle. The others are odd reflections. It can be shown that continuing this process of reflections indefinitely results in a cover of the open unit disc.

The rest of the idea is the same. One uses Carathéodary's version of the Riemann Mapping theorem to get a homemorphism from the inner most triangle (the original one) to $\overline{\mathfrak{h}}$ which takes the boundary of the triangle to $\mathbf{R} \cup \infty$ and the map on the interior is univalent surjective, and finally, we can ensure that ω^2 maps to 0, 1 maps to 1, and ω to ∞. Using Schwarz's reflection principle over and over again, we obtain a holomorphic function

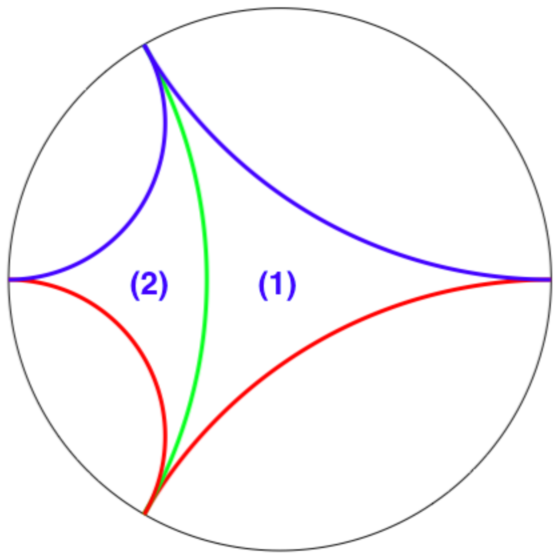
$$\lambda^* \colon \Delta \to \mathbf{C} \smallsetminus \{0, 1\}.$$

This map is a covering map. It follows that any entire function $f \colon C \to \mathbf{C} \smallsetminus \{0, 1\}$ lifts to a entire function $\widetilde{f} \colon \mathbf{C} \to \Delta$ in such a way that $f = \lambda^* \circ \widetilde{f}$. By Liouville's theorem, $\widetilde{f}$ is a constant, whence so is f.

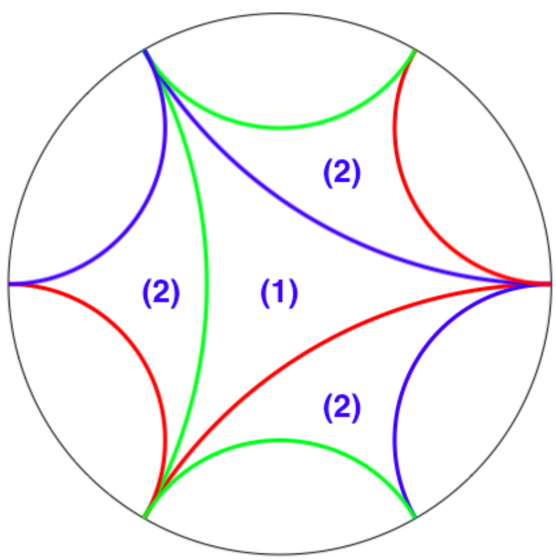
Here is the development of the reflections. We begin with the triangle below. The labelled points are the cube roots of unity. The interior of the triangle has been labelled (1) to indicate it is the first generation of triangles. Note that the angles of the triangle are zero, since each edge meets the unit circle in a perpendicular fashion.



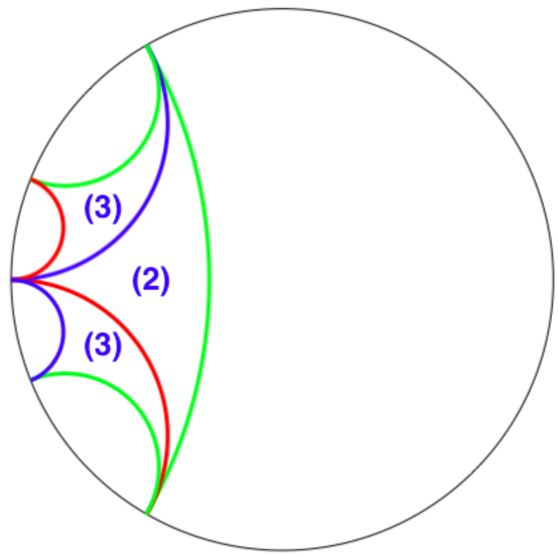
The second generation of triangles is developed in the following manner. Reflecting about the green arc gives:



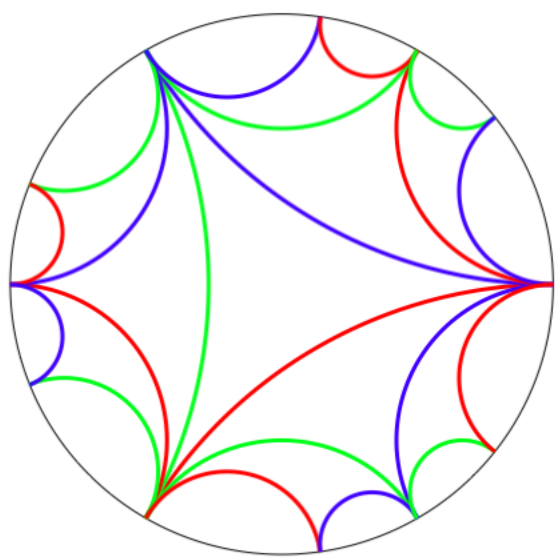
Doing this over the red and blue arcs yields all the second generation triangles arising from the original (first generation) triangle.



Consider the second generation triangle which lies to the left of the original green arc, and reflect about the the two new arcs of the triangle. One obtains:



Doing this to other triangles labelled (2) we get all the triangles up to and including the third generation triangles.



It might be an idea to work out further reflections (the fourth generation of reflections). Also it may be helpful to shade all the even generation triangles in one colour and the odd generation triangles in another colour.