LECTURE 25

Date of Lecture: April 12, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

The unit circle will be denoted $\mathbf{T}$ instead of C. As usual Δ will denote the open unit disc. The Riemann sphere will be denoted $\mathbb{P}^1$.

1. Meromorphic functions revisited

1.1. **Generalities.** Please refer to Problem 4(b) of HW 5 for this discussion. First, note that the domain of the meromorphic function $\alpha(z) = \dfrac{1}{z}$ can be extended to $\mathbb{P}^1$ in a continuous way by setting $\alpha(\infty) = 0$. Thus we have a homeomorphism

$$\alpha\colon \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$$

which we continue to write as $\alpha(z) = \dfrac{1}{z}$.

A meromorphic function $f(z)$ on an open set U of $\mathbf{C}$ can be regarded as a continuous map

$$f\colon U \to \mathbb{P}^1$$

such that $S = f^{-1}(\infty)$ is a discrete subset of U, with $f|_{U\smallsetminus S}$ holomorphic holomorphic. Note that f has poles at the isolated singularities, namely the points of S.

If U is the complement of a compact set in $\mathbf{C}$ then U defines an open neighbourhood of ∞, i.e., $\widehat{U} := U \cup \{\infty\}$ is an open neighbourhood of ∞. Now $V = \alpha(\widehat{U})$ is an open neighbourhood of 0 and we say f is meromorphic on $\widehat{U}$ if $f\circ\alpha$ is meromorphic on $V \smallsetminus \{\infty\}$. We say f has a pole at ∞ if $f\circ\alpha$ has a pole at 0, and it has a removable singularity at ∞ if $f\circ\alpha$ has a removable singularity at 0. In the latter case, clearly f extends in a continuous manner to $\widehat{U}$ with $f(\infty)$ being defined as the value of $f\circ\alpha$ at 0 after the removable singularity has in fact been removed.

Note that a non-constant meromorphic function must be an open map to $\mathbb{P}^1$. This follows in a straightforward way from the above definitions and from the open mapping theorem for holomorphic functions. For example, in a neighbourhood of a pole of f in $\mathbf{C}$, $\alpha\circ f$ is holomorphic, and hence open, and α is a homeomorphism. If ∞ is a pole of f then one uses $\alpha\circ f\circ\alpha$ to draw the same conclusion.

It therefore makes sense to talk about meromorphic functions on $\mathbb{P}^1$. From the above discussion these are continuous functions

$$f\colon \mathbb{P}^1 \to \mathbb{P}^1$$

such that $S = f^{-1}(\infty)$ is finite (discrete subsets of the Riemann sphere are finite since they are compact and discrete), and such that f is holomorphic on $\mathbf{C}\smallsetminus S$ and has a removable singularity or a pole at ∞. According to Problem 4(b) of HW 5, these are exactly the *rational functions*, i.e., functions of the form

$$f(z) = \frac{p(z)}{q(z)}$$

where p and q are polynomials, and q is a non-zero polynomial. After reducing the situation where $p(z)$ and $q(z)$ have no common factors, the poles of f in $\mathbf{C}$ occur where q vanishes. The value of f at ∞ is obtained by the standard tricks for obtaining limits of rational functions at infinity. This yields the following. If $\deg p > \deg q$ then f has a pole of order $\deg p - \deg q$ at ∞. If $\deg p < \deg q$, f has a zero of order $\deg q - \deg p$ at ∞. If $\deg p = \deg q$ then $f(\infty) = a/b$ where a is the leading coefficient of p and b is the leading coefficient of q.

1.2. **Linear Fractional Transformations.** We now examine meromorphic functions

$$T\colon \mathbb{P} \to \mathbb{P}$$

which are one-to-one. Since non-constant meromorphic functions are open maps, T is necessarily onto. There can only be one point $w^* \in \mathbb{P}^1$ such that $T(w^*) = \infty$. Write $T(z) = p(z)/q(z)$ where p and q are co-prime polynomials with q monic. If $w^* \in \mathbf{C}$ then $q(z) = z - w^*$ and $(z - w^*)T(z)$ is entire with at most a simple pole at ∞. By Problem 3 of HW 3, $T(z)$ is either a constant or a linear polynomial. Thus $T(z) = (az + b)/(z - w^*)$ where $a = 0$ only if $(z - w^*)T(z)$ is constant. Since the numerator and denominator are co-prime, we must have $-aw^* - b \neq 0$. If, on the other hand $w^* = \infty$, then T restricts to an entire function on $\mathbf{C}$, with a simple pole at ∞ and hence must be of the form $T(z) = az + b$ with $a \neq 0$. In either case we have

$$T(z) = \frac{az + b}{cz + d} \tag{1.2.1}$$

with $a, b, c, d \in \mathbf{C}$ satisfying $ad - bc \neq 0$.

Definition 1.2.2. A meromorphic function $T\colon \mathbb{P}^1 \to \mathbb{P}^1$ is called a *linear fractional transformation* or a *Möbius transformation* if it is of the form (1.2.1) with $ad - bc \neq 0$.

Linear fractional transformations are exactly the same as meromorphic functions on $\mathbb{P}^1$ which are one-to-one.

Here are some observations to help you familiarise yourself with all this. In what follows T is as in (1.2.1) with the accompanying condition $ad - bc \neq 0$.

1) Consider the two cases $c = 0$ and $c \neq 0$. If $c = 0$ then $T(z) = \alpha z + \beta$ with $\alpha \neq 0$. In other words T restricted to $\mathbf{C}$ is non-constant and entire. In this case $T(\infty) = \infty$. If on the other hand $c \neq 0$, then T has a simple pole at $-d/c$, T is holomorphic on $\mathbf{C} \smallsetminus \{-d/c\}$ and $T(\infty) = a/c$.

2) Since linear fractional transformations are the same as one-to-one meromorphic functions on $\mathbb{P}^1$, it follows that they form a group under composition. In particular *if T is a linear fractional transformation so is T^{-1}.*

3) Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Since $ad - bc \neq 0$, A is non-singular. We sometimes write $T = T_A$ to show the dependence of T on A, and this notation is useful in what follows. Let B be another invertible matrix. Then it is easy to check that

$$T_B \circ T_A = T_{BA}. \tag{1.2.3}$$

In other words, if $\mathbf{GL}(2,\, \mathbf{C})$ is the group of 2×2 non-singular matrices, we have an action of on $\mathbb{P}^1$:

$$\mathbf{GL}(2,\, \mathbf{C}) \times \mathbb{P}^1 \to \mathbb{P}^1.$$

What (1.2.3) says is that $A \mapsto T_A$ *is a group homomorphism from* $\mathbf{GL}(2,\, \mathbf{C})$ *to the group of linear fractional transformations. In particular* $T_A^{-1} = T_{A^{-1}}$.

Theorem 1.2.4. *Let T be a linear fractional transformation. Suppose there exist three distinct points z_1, z_2, and z_3 on $\mathbb{P}^1$ such that $T(z_i) = z_i$ for $i = 1, 2, 3$. Then T is the identity map.*

Proof. Let $T = T_A$ where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a non-singular matrix. First assume $z_i \in \mathbf{C}$ for all i. Then each of the z_i's satisfies the quadratic equation

$$cz^2 + (d - a)z - b = 0.$$

Since there are three distinct solutions to this equation, all the coefficients must vanish, i.e., $a = d$, and $c = b = 0$. Note that $a = d$ is non-zero since A is invertible. It follows that $T(z) = z$ for all $z \in \mathbf{C}$ and hence for all $z \in \mathbb{P}^1$.

Now suppose one of the z_i's is equal to ∞, say z_1. Then $T(\infty) = \infty$, which means T restricted to $\mathbf{C}$ is entire, i.e, $T(z) = az + b$ and $c = 0$, $d = 1$. Note that since $\det A = a$ in this case, therefore $a \neq 0$ (in any case, T is one-to-one and hence cannot be a constant, and so $a \neq 0$). The equation $T(z) = z$ translates to

$$(a - 1)z + b = 0$$

and this has two distinct solutions. It follows that the coefficients vanish, i.e., $a = 1$ and $b = 0$. Thus in this case too $T(z) = z$ for all z. $\square$

Example 1.2.5. Suppose $T\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a one-to-one meromorphic function. We know that T must be a linear fractional transformation, say $T = T_A$ where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. How do we find A (which is unique up to scalar multiplication) based on some qualitative knowledge of T? The strategy is as follows. Let $r_1 = T^{-1}(0)$, $r_2 = T^{-1}(1)$, and $r_3 = T^{-1}(\infty)$. If Q is any linear fractional transformation such that $Q(r_1) = 0$, $Q(r_2) = 1$ and $Q(r_3) = \infty$, then by Theorem 1.2.4, $Q = T$. Here are the cases to consider:

(a) Suppose all the r_i lie in $\mathbf{C}$ (i.e., $\infty \notin \{r_1, r_2, r_3\}$). Then then one sees easily from the above discussion that

$$T(z) = \frac{r_2 - r_3}{r_2 - r_1} \frac{z - r_1}{z - r_3}. \tag{1.2.6}$$

If we set

$$R := \frac{r_2 - r_3}{r_2 - r_1}$$

then $T = T_A$, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $a = R$, $b = -Rr_1$, $c = 1$, and $d = -r_3$. We also have

$$\det A = R(r_1 - r_3) = \frac{r_2 - r_3}{r_2 - r_1}(r_2 - r_1). \tag{1.2.7}$$

Note that since the r_i are distinct, the ratio on the right must be non-zero, as expected.

(b) Suppose $r_1 = \infty$. Then by checking the values of T at r_1, r_2 and r_3 and using Theorem 1.2.4 we see that

$$T(z) = \frac{r_2 - r_3}{z - r_3}. \tag{1.2.8}$$

Clearly

$$\det A = r_3 - r_2. \tag{1.2.9}$$

(c) Suppose $r_2 = \infty$. Then

$$T(z) = \frac{z - r_1}{z - r_3}, \tag{1.2.10}$$

and

$$\det A = r_1 - r_3. \tag{1.2.11}$$

(d) Finally, suppose $r_3 = \infty$. Then

$$T(z) = \frac{z - r_1}{r_2 - r_1}, \tag{1.2.12}$$

and

$$\det A = r_2 - r_1. \tag{1.2.13}$$

We point out that just as in case (a), in all the other cases $\det A$ can be seen to be non-zero because the r_i are distinct.

2. The unit disc and the upper half plane

2.1. **Riemann Mapping at the boundary.** We have seen that if Ω is a simply connected bounded region with every boundary point simple, then every univalent surjective map $f\colon \Omega \to \Delta$ extends uniquely to a continuous one-to-one onto map $f\colon \overline{\Omega} \to \overline{\Delta}$. We will call a map $f\colon \overline{\Omega} \to \overline{\Delta}$ a Riemann map if $f|_{\partial\Omega}$ takes values in Δ, and this restriction is univalent with $f(\Omega) = \Delta$.

Lemma 2.1.1. *Let Ω be a bounded simply connected region whose boundary consists of simple points, and suppose $f_i\colon \overline{\Omega} \to \overline{\Delta}$, $i = 1, 2$, are two Riemann maps such that $f_1|_{\partial\Omega} = f_2|_{\partial\Omega}$. Then $f_1 = f_2$.*

Proof. Let $g_i = f_i^{-1}$, $i = 1, 2$. It is enough to show $g_1 = g_2$. From the hypothesis, $g_1|_{\mathbf{T}} = g_2|_{\mathbf{T}}$. Now $g_i|_{\mathbf{T}}$ is the radial limit function of g_i, whence $g_1 = g_2$. $\square$

2.2. **Automorphisms of the unit disc.** Suppose $S\colon \Delta \to \Delta$ is biholomorphic. We have seen, using Schwarz's Lemma that S must be of the form

$$S(z) = e^{i\theta}\frac{z + b}{1 - \bar{b}z} \tag{2.2.1}$$

with $b \in \Delta$ and hence S extends to $\overline{\Delta}$ in a continuous bijective way. In particular $S|_{\mathbf{T}}$ is homeomorphism from $\mathbf{T}$ to itself. Note that S is a linear fractional transformation, and hence extends to an automorphism of $\mathbb{P}^1$.

We will show later the following: Let ζ_1, ζ_2, ζ_3 be three distinct points on $\mathbf{T}$ such that for $i \in \{1, 2, 3\}$, the arc from ζ_i to ζ_{i+1} which does not contain ζ_{i-1} (with $\zeta_4 = \zeta_1$ and $\zeta_0 = \zeta_3$) is in the counter-clockwise direction. Then there exists an automorphism S of Δ, which when extended to $\overline{\Delta}$, satisfies $S(\zeta_1) = -i$, $S(\zeta_2) = i$, and $S(\zeta_3) = 1$. This can be deduced from (2.2.1), but is perhaps best seen by transferring the problem to the upper half-plane $\mathfrak{h}$, and we indicate how below.

2.3. **Automorphisms of the upper half-plane.** As before, let $\mathbb{P}^1$ be the Riemann sphere, with $\mathbf{C}$ regarded as an open subset, namely $\mathbf{C} = \mathbb{P}^1 \setminus \{\infty\}$. Consider the Cayley transformation

$$\phi \colon \mathbb{P}^1 \to \mathbb{P}^1$$

given by the linear fractional transformation

$$\phi(z) = -i\frac{z+i}{z-i}. \tag{2.3.1}$$

It is easy to see that ϕ takes points in $\mathbf{C}$ which lie "outside" $\mathbf{T}$ to the upper half-plane $\mathfrak{h}$, and the points which lie "inside" $\mathbf{T}$ to the lower half-plane, $\mathfrak{h}^-$. Indeed $\phi(\mathbf{T}) \subset \mathbf{R} \cup \{\infty\}$ and hence $\phi(\mathbf{T}) = \mathbf{R} \cup \{\infty\}$ since ϕ is a homeomorphism. Thus either $\phi(\mathfrak{h}) = \mathfrak{h}$ or $\phi(\mathfrak{h}) = \mathfrak{h}^-$. Since $\phi(0) = i$, the assertion is proven.

Note that under ϕ, $\mathbf{T}$ maps homeomorphically onto $\mathbf{R} \cup \{\infty\}$. Every automorphism of $\mathfrak{h}$ is necessarily of the form

$$T = \phi \circ S \circ \phi^{-1} \tag{2.3.2}$$

where S is an automorphism of Δ, and we know that such an S must necessarily take the form (2.2.1) and hence is a linear fractional transformation. It follows from (2.3.2) that T is a linear fractional transformation.

Since T is a linear fractional transformation, it is defined on all of $\mathbb{P}^1$. From our discussion on Δ, it is clear that $T(\mathbf{R} \cup \{\infty\}) = \mathbf{R} \cup \{\infty\}$. By Theorem 1.2.4 if 0, 1 and ∞ are fixed by T, then T must be the identity map. For this reason, if we know the points r_1, r_2, r_3 in $\mathbf{R} \cup \{\infty\}$ such that $T(r_1) = 0$, $T(r_1) = 1$, and $T(r_3) = \infty$, then we know T. In fact the discussion in Example 1.2.5 applies with the r_i being restricted to the circle $\mathbf{R} \cup \{\infty\}$ in the Riemann sphere.

From the discussion in Example 1.2.5, since the $r_i \in \mathbf{R} \cup \{\infty\}$, we see that $T = T_A$ where the entries in A are real, i.e., $A \in \mathbf{GL}(2, \mathbf{R}) \subset \mathbf{GL}(2, \mathbf{C})$.

It is worth asking, suppose we have a linear fractional transformation $T = T_A$, where A is in $\mathbf{GL}(2, \mathbf{R})$ (instead of allowing it range over the larger set $A \in \mathbf{GL}(2, \mathbf{C})$), then when does it give an automorphism of $\mathfrak{h}$? We have

$$T(z) = \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbf{R}$. It is clear that $T(\mathbf{R} \cup \{\infty\}) = \mathbf{R} \cup \{\infty\}$. Now the complement of $\mathbf{R} \cup \{\infty\}$ in $\mathbb{P}^1$ is the union of two disjoint open sets, $\mathfrak{h}$ and $\mathfrak{h}^-$. Since T is a (topological) automorphism of $\mathbb{P}^1$ we must have either $T(\mathfrak{h}) = \mathfrak{h}$ or $T(\mathfrak{h}) = \mathfrak{h}^-$. We are interested in automorphisms of $\mathfrak{h}$ and for these it is the first option we need. To check that $f(\mathfrak{h}) = \mathfrak{h}$ it is enough that $f(i) \in \mathfrak{h}$. From the expression for $T(z)$ above, it is clear that

$$f(i) = \frac{ac + bd + i(ad - bc)}{|d + ic|^2}.$$

It follows that $f(i) \in \mathfrak{h}$ if and only if $ad - bc > 0$. We have thus proven the following

Theorem 2.3.3. *The automorphism group of $\mathfrak{h}$, i.e., the group of univalent surjective maps $\mathfrak{h} \to \mathfrak{h}$ with composition of maps as the group operation, is the group of linear fractional transformations*

$$T(z) = \frac{az+b}{cz+d}$$

such that $a, b, c, d \in \mathbf{R}$ and $ad - bc > 0$.

Let r_i, $i \in \{1, 2, 3\}$ be three distinct points in $\mathbf{R} \cup \{\infty\}$. The ordered triple of points (r_1, r_2, r_3) defines an orientation on the circle $\mathbf{R} \cup \{\infty\}$, namely the one obtained from travelling first from r_1 to r_2 along the arc which does not contain r_3, then moving from r_2 to r_3 along the arc which does not contain r_1 and finally moving from r_3 to r_1 in the arc between r_3 to r_1 which does not contain r_2. Two observations are worth making:

- Every cyclic permutation of (r_1, r_2, r_3) defines the same orientation of $\mathbf{R} \cup \{\infty\}$. Write $[r_1, r_2, r_3]$ for the equivalence class of triples which are cyclic permutations of (r_1, r_2, r_3).
- The orientation defined by $\sigma_0 = [0,\, 1,\, \infty]$ is such that $\mathfrak{h}$ falls to left as one moves along $\mathbf{R}$ in the directions defined by σ. We call the orientation given by σ_0 the *positive orientation*.

Theorem 2.3.4. *Let $r_1, r_2, r_3 \in \mathbf{R} \cup \{\infty\}$ be three distinct points, $\mathbf{r} = (r_1,\, r_2, r_3)$ and $T_{\mathbf{r}}$ the linear fractional transformation such that $r_1 \mapsto 0$, $r_2 \mapsto 1$, and $r_3 \mapsto \infty$ as in Example* 1.2.5*. Then $T_{\mathbf{r}}$ defines an automorphism of $\mathfrak{h}$ if and only if $[r_1,\, r_2,\, r_3]$ defines the same orientation of $\mathbf{R} \cup \{\infty\}$ as σ_0.*

Proof. Let $\sigma = [r_1,\, r_2,\, r_3]$. We remind the reader that $T_{\mathbf{r}}$ is a continuous (in fact C^∞) one-to-one map from $\mathbb{P}^1$ to itself. Topological considerations show that if $T_{\mathbf{r}}(\mathfrak{h}) = \mathfrak{h}$ then the orientation defined by σ is the same as that of σ_0. In other words the "only if" part is true.

On the other hand if σ has the same orientation as σ_0 then one has the following cases.

(a) Suppose $r_i \in \mathbf{R}$ for $i = 1, 2, 3$, then the only possibilities are (i) $r_1 < r_2 < r_3$, (ii) $r_2 < r_3 < r_1$, (iii) $r_3 < r_1 < r_1$. According to Theorem 2.3.3 and (1.2.7), we have to check that
$$\frac{r_2 - r_3}{r_2 - r_1}(r_2 - r_1) > 0.$$
This is easily seen to be true in each of the cases (i), (ii), and (iii).

(b) Suppose $r_1 = \infty$. For σ to have positive orientation, the only possibility is $r_2 < r_3$. According to Theorem 2.3.3 and (1.2.9) we have to check that $r_3 - r_2 > 0$, which is clearly true in this case.

(c) Suppose $r_2 = \infty$. Then σ has positive orientation if and only if $r_3 < r_1$. From (1.2.11) and Theorem 2.3.3 we are done in this case too.

(d) Suppose $r_3 = \infty$. Then σ has positive orientation if and only if $r_1 < r_2$. We appeal to (1.2.13) and Theorem 2.3.3. The result follows.

□

Remark 2.3.5. Let Ω be a bounded simply-connected region with simple boundary points, such that the boundary $\partial\Omega$ is a piecewise smooth curve. Note that by Carathéodary's extension of the Riemann Mapping Theorem, $\partial\Omega$ is homeomorphic to $\mathbf{T}$ and by the Cayley transform, to $\mathbf{R} \cup \{\infty\}$. Let α, β, γ be three distinct boundary points of Ω such that when one moves from α to β along the boundary segment which does not meet γ, and then from β to γ along the boundary segment which does not meet α and finally from γ to α along the remaining boundary segment, then Ω lies to the left. Under these hypotheses, if $f\colon \Omega \to \mathfrak{h}$ is a univalent surjective map and f also denotes the continuous extension $\overline{\Omega} \to \overline{\mathfrak{h}} = \mathfrak{h} \cup \mathbf{R} \cup \{\infty\}$ then, $[f(\alpha),\, f(\beta),\, f(\gamma)]$ gives positive orientation to $\mathbf{R} \cup \{\infty\}$. The assertions are obvious, since conformal holomorphic maps are orientation preserving.

3. **Reflections**

3.1. **Reflection about a line.** Let L be a line in the complex plane. Define the reflection with respect to L to be the map

$$R_L\colon \mathbf{C} \to \mathbf{C}$$

as the map $z \mapsto z^*$ where z^* is obtained by drawing a line through z perpendicular to L and marking off z^* on this perpendicular line at an equal distance from L as z in a way that $z \neq z^*$ unless $z \in L$, in which case $z = z^*$. In terms of formulas, if θ is the angle in $[-\pi, \pi)$ that L makes with the positive real axis, and z_0 is a fixed point in L, then

$$z^* = R_L(z) = e^{i\theta}\overline{e^{-i\theta}(z - z_0)} + z_0 = e^{2i\theta}(z - z_0).$$

The map R_L can be extended to $\mathbb{P}^1$ by setting $R_L(\infty) = \infty$. Thus

$$R_L\colon \mathbb{P}^1 \to \mathbb{P}^1.$$

3.2. **Reflection about the unit circle.** Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Cayley transform (2.3.1). Define the reflection with respect to $\mathbf{T}$ to be the map

$$R_{\mathbf{T}}\colon \mathbb{P}^1 \to \mathbb{P}^1$$

given by $z \mapsto z^*$ where

$$z^* = \phi^{-1}\overline{\phi(z)}$$

where the conjugate of ∞ is taken to be ∞. If $z \in T$, note that $z^* = z$. Otherwise, z^* is inside $\mathbf{T}$ if z is outside T, and z^* is outside $\mathbf{T}$ when z is inside T.

3.3. **Reflection about an arbitrary circle.** Let C be a circle of radius ρ, $0 < \rho < \infty$ and with centre z_0. Define the reflection with respect to C to be the map

$$R_C\colon \mathbb{P}^1 \to \mathbb{P}^1$$

given by

$$R_C(z) = z_0 + \rho R_{\mathbf{T}}\Big(\frac{z - z_0}{\rho}\Big).$$

Note once again that $R_C(z) = z$ if and only if $z \in C$. If z lies "outside" C, then $R_C(z)$ lies inside C, and vice-versa.

3.4. **The Schwarz Reflection Principle for line segments and circular arcs.** The following is straightforward:

Lemma 3.4.1. *Let L be a line or a circular arc. Suppose Ω is a region in $\mathbf{C}$ such that $\Omega \cap L = \emptyset$. Let $f\colon \Omega \to \mathbf{C}$ be holomorphic. Then the function*

$$g\colon R_L(\Omega) \to \mathbf{C}$$

given by

$$g(z) = \overline{f(R_L(z))}$$

is holomorphic.

Definition 3.4.2 (Symmetric region)**.** Let L be a line or a circle in $\mathbf{C}$. A region Ω is called *symmetric* with respect to L if $\Omega = R_L(\Omega)$.

Note that if L is a straight line or a circle, then $\mathbf{C} \smallsetminus L$ has two connected components. If the components are U_1 and U_2 then $R_L(U_1) = U_2$ and $R_L(U_2) = U_1$.

The following is an obvious re-phrasing of the usual Schwarz Reflection Principle.

Theorem 3.4.3 (The Schwarz Reflection Principle)**.** *Let Ω be a region in* $\mathbf{C}$ *which is symmetric with respect to L, where L is a line or a circle. Let U_1 and U_2 be the connected components of $\mathbf{C} \smallsetminus L$, and $\Omega_1 = \Omega \cap U_1$, $\Omega_2 = \Omega \cap U_2$ and $M = \Omega \cap L$. Suppose $f\colon \Omega_1 \to C$ is a holomorphic map such that if $\{z_n\}$ is a sequence in Ω_1 converging to a point in M then $\lim_{n\to\infty} \mathrm{Im}(f(z_n)) = 0$. Then f extends to a holomorphic function on Ω. The extended function satisfies the relation*

$$f(z) = \overline{f(R_L(z))} \qquad (z \in \Omega).$$

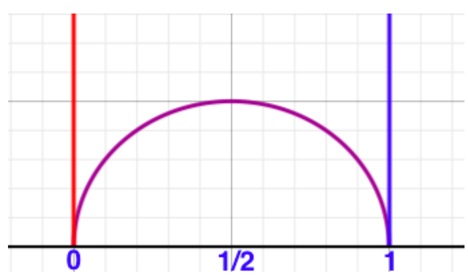
In particular f is real-valued on M.

4. **Picard's Modular Function**

4.1. Let Ω be the region in the upper half-plane $\mathfrak{h}$ given by

$$\Omega = \left\{ z \in \mathfrak{h} \mid 0 < \mathrm{Re}\,(z) < 1, \left| z - \frac{1}{2} \right| > \frac{1}{2} \right\}.$$

The region is depicted below. The boundary of Ω consists of the vertical line $\mathrm{Re}\,(z) = 0$ on the left (the red vertical line), the line $\mathrm{Re}\,(z) = 1$ on the right (the blue vertical line), and the semi-circle $|z - 1/2| = 1/2$, $\mathrm{Im}\,(z) \geq 0$ at the bottom (coloured purple).



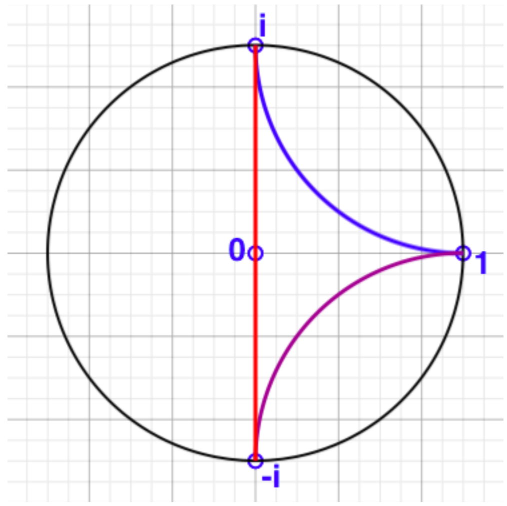
In $\mathbb{P}^1$, the boundary would be all this along with the point ∞ which is the intersection of the two vertical lines mentioned, in the Riemann sphere. Thus on $\mathbb{P}^1$ the boundary of Ω is a triangle. This can be brought to the finite plane—in fact to $\overline{\Delta}$—by the transformation

$$T(z) = \phi^{-1}(2z - 1)$$

where ϕ is the Cayley map (2.3.1). Note that

$$\phi^{-1}(z) = i\frac{z - i}{z + i}.$$

The boundary of Ω in $\mathbb{P}^1$ transforms to the boundary of $T(\Omega)$ which is a triangle with vertices $\{-i, i, 1\}$ with some edges being circular arcs. Indeed $T(0) = \phi^{-1}(-1) = -i$, $T(1) = \phi^{-1}(1) = 1$, and $T(\infty) = \phi^{-1}(\infty) = i$. The transformed region is the one depicted below.

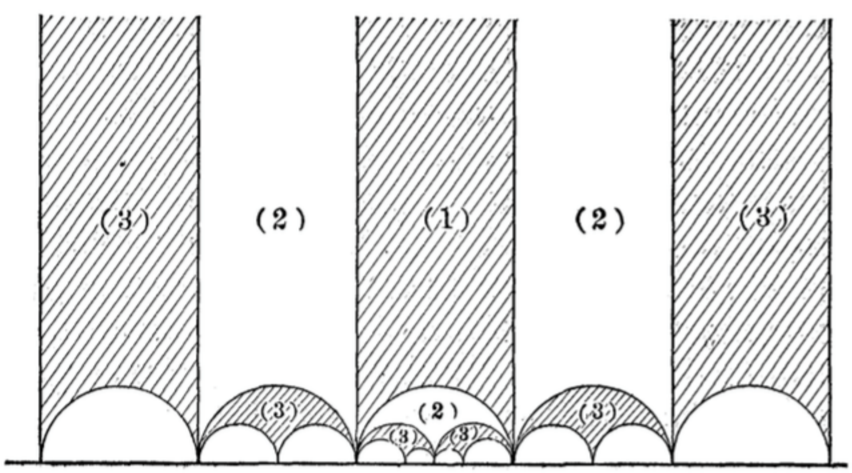


$T(\Omega)$ is the region bounded by the red line on the left and by the blue and purple circular arcs. The colours correspond to the colours used on the boundary of Ω. For the curious, the blue arc is part of the circle $(x-1)^2 + (y-1)^2 = 1$ and the purple arc is part of the circle $(x-1)^2 + (y+1)^2 = 1$. It is a good exercise to see that under T, the boundary of Ω in $\mathbb{P}^1$ does transform to what is depicted in the picture. Carathéodary's version of the Riemann Mapping Theorem applies to $T(\Omega)$ and hence to Ω.

Translating the above reasoning to $\mathfrak{h}$, using the Cayley map ϕ, we see that if $S^*\colon \mathfrak{h} \to \mathfrak{h}$ is a biholomorphism, it extends continuously to the boundary $\mathbf{R} \cup \{\infty\}$ and if the extended map fixes 0, 1, and ∞, it is the identity map (see Theorem 1.2.4).

By Remark 2.3.5 and Theorem 2.3.4 we therefore have a unique univalent surjective map $\lambda\colon \Omega \to \mathfrak{h}$ such that it extends uniquely to a continuous map on the boundary taking 0 to 0, 1 to 1, and ∞ to ∞.

In what follows we use the term region to include boundaries – we will only do it for this discussion. Consider the picture given below (taken from page 62 of Osgood's 1898 lectures [0]).



The interior of the region labelled (1) is Ω. That region can be reflected, via reflections with respect to lines and circles, into three regions, namely the regions labelled (2). Since λ is real on the boundary of Ω, Schwarz's Reflection Principle says that λ can be extended to each of the regions labelled 2, so that the union of the regions labelled (1) and (2) together with their common boundaries allow for the extension of λ. Note that when λ extends from (1) to the region on its immediate right, the range of λ is $\mathbf{C} \smallsetminus (\infty, 0]$. Similarly when λ is reflected to the only bounded region labelled (2), the extended function takes values in $\mathbf{C} \smallsetminus [0, 1]$, and when we reflect on the left, the range is $\mathbf{C} \smallsetminus [1, \infty)$. Overall, the range of λ in the interior of the union of the regions labelled (1) and (2) is $\mathbf{C} \smallsetminus \{0, 1\}$. We can continue this process. Each of the regions labelled (2) have two circular arcs or straight lines through which further refections can be effected, and this gives six regions labelled (3). If one continues this process indefinitely, the interior of the union of all possible reflections is the upper half-plane $\mathfrak{h}$, and we get a holomorphic map, the *Picard modular function*,

$$\lambda\colon \mathfrak{h} \to \mathbf{C} \smallsetminus \{0,\, 1\}. \tag{4.1.1}$$

It is easy to see that this is a covering map. Indeed, if $z_0 \in \mathfrak{h}$ then pick an open ball B_0 with z_0 as centre lying wholly in $\mathfrak{h}$, and we have a unique open set U_0 in Ω mapping to it under λ biholomorphically. The inverse image of B_0 is the union of all the even reflections of U_0, and these are clearly pairwise disjoint, and each of them maps biholomorphically on to B_0. Similar considerations take care of matters when $z_0 \in \mathfrak{h}^-$. If z_0 lies on the boundary, say on $(1, \infty)$, pick the point w_0 on the line $\mathrm{Re}\,(z) = 1$ which corresponds to it under λ. Pick a ball B_0 with centre z_0 such that $B_0 \cap \mathbf{R} \subset (1, \infty)$. Then we have an open neighbourhood U_0 of w_0 lying entirely in two regions adjoining w_0, which maps biholomorphically on to B_0. Once again all even reflections of U_0 form the inverse image of B_0 under λ and these are clearly disjoint and biholomorphic, under λ, to B_0. The same considerations work $z_0 \in (0, 1)$ and in $(\infty, 0)$. We thus have (since $\mathfrak{h}$ is obviously simply connected)

Theorem 4.1.2. *The Picard modular function $\lambda\colon \mathfrak{h} \to \mathbf{C} \smallsetminus \{0,\, 1\}$ given in* (4.1.1) *is a holomorphic covering map. The map λ is the universal cover of $\mathbf{C} \smallsetminus \{0,\, 1\}$*

This gives us

Theorem 4.1.3 (Picard's Little Theorem)**.** *Let a and b be two distinct points of $\mathbf{C}$ and suppose f is an entire function which does not take values in $\{a,\, b\}$. Then f is a constant.*

Proof. By replacing $f(z)$ by $\dfrac{f(z) - a}{b - a}$ if necessary, we assume without loss of generality that $\{a,\, b\} = \{0,\, 1\}$. Thus

$$f\colon \mathbf{C} \to \mathbf{C} \smallsetminus \{0,\, 1\}.$$

Let $z_0 \in \mathbf{C}$ and pick $\zeta_0 \in \mathfrak{h}$ such that $f(z_0) = \lambda(\zeta_0)$. Since f is continuous and λ is the universal covering space of $\mathbf{C} \smallsetminus \{0,\, 1\}$, there exists a unique continuous map

$$\widetilde{f}\colon \mathbf{C} \to \mathfrak{h}$$

such that $\widetilde{f}(z_0) = \zeta_0$ and $\lambda \circ \widetilde{f} = f$. It is easy to see that $\widetilde{f}$ must be holomorphic. If an entire function takes values in $\mathfrak{h}$ it must be a constant by Lousivlle's theorem, since $\mathfrak{h}$ is biholomorphic to Δ by (say) the Cayley transformation (2.3.1). This means f is constant. $\square$

References

[0] W.F. Osgood, *Selected topics in the general theory of functions.* Six lectures delivered before the Cambridge Colloquium, August 22-27, 1898. AMS Bull. 5:59-87.