LECTURE 24

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Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

The unit circle will be denoted $\mathbf{T}$ instead of C. As usual Δ will denote the open unit disc. As before, for $1 \le p \le \infty$ we identify $L^p([-\pi,\, \pi])$ with $L^p(\mathbf{T})$, where the σ-algebras in play are the Borel σ-algebras and the measures, the normalised Lebesgue on $[-\pi,\, \pi]$ (the usual Lebesgue measure divided by 2π) and the Haar measure on $\mathbf{T}$. The identification is the standard one.

1. Boundary values of the Riemann mapping

This is a rich topic. We will confine ourselves to one aspect, namely Caratheodary's beautiful theorem that if $f\colon \Omega \to \Delta$ is a univalent surjective map (note this forces Ω to be simply connected) then f can be extended as a continuous function to simple points of the boundary. This theorem makes it easier to prove Picard's theorem,but is clearly very interesting in its own right.

1.1. Simple boundary points.

Let Ω be a simply connected region. Recall that the boundary of Ω is $\partial\Omega = \overline{\Omega} \cap (\mathbf{C} \smallsetminus \Omega)$. Points of $\partial\Omega$ are called boundary points. Recall that a *curve* in Ω is a continuous map $\gamma\colon [a,\, b] \to \Omega$ (see beginning of § 2 of Lecture 2).

Definition 1.1.1 (Simple boundary point)**.** A boundary point β of Ω is called a *simple boundary point* if it has the following property. Given a sequence $\{z_n\}$ of points in Ω such that $z_n \to \beta$ as $n \to \infty$, there exists a curve $\gamma\colon [0,1] \to \overline{\Omega}$ such that $\gamma(t) \in \Omega$ for $t < 1$, $\gamma(1) = \beta$ and there exist a sequence $\{t_n\}$ with $0 < t_1 < \dots < t_n < t_{n+1} < \dots$ such that $\gamma(t_n) = z_n$ for every $n \in \mathbb{N}$.

Examples 1.1.2. Here are two examples:

1) Let $\Omega = \Delta \smallsetminus [0,1)$. Then $\beta = 1/2$ is a boundary point of Ω. It is easy to see that β cannot be a simple boundary. Indeed, consider the sequence $z_n = 1/2 + (-1)^n(1/n)i$. Then $z_n \to \beta$. It is not possible to connect the $z_n's$ by a contour $\gamma\colon [0,1] \to \overline{\Omega}$ in such a way that $\gamma(t) \in \Omega$ for $t < 1$ and $\gamma(1) = \beta = 1/2$. In fact, as is easy to see, every β in the open interval $(0,1)$ provides an example of a non-simple boundary point of Ω.

2) Suppose a boundary point β is such that $B(\beta, r) \cap \Omega$ is connected for all sufficiently small radii r. Then a little thought shows that β must be a simple boundary point of Ω. Indeed, we have a subsequence $\{z_{n_k}\}$ of z_n such that $|\beta - z_m| < 1/k$ for all $m \ge n_k$. For sufficiently large k, say $k \ge k_0$, our hypothesis implies that $B(\beta,\, 1/k) \cap \Omega$ is connected. Connect z_m to z_{m+1} by any curve in Ω for $m \le k_0$. For $m \in \{n_k, \dots, n_{k+1} - 1\}$, with $k \ge k_0$, connect z_m to z_{m+1} by a curve in $B(\beta,\, 1/k) \cap \Omega$ (which is connected). Take the amalgamation of all the curves, to get a map $\gamma\colon [0,1) \to \Omega$ such that $\lim_{t\to 1} \gamma(t) = \beta$. Extend γ to $[0,\, 1]$ by setting $\gamma(1) = \beta$.

Theorem 1.1.3 (Carathéodary)**.** *Let $f\colon \Omega \to \Delta$ be a univalent surjective map from a bounded simply connected region Ω.*

(a) *If β is a simple boundary point of Ω then f can be extended as a continuous function to $\Omega \cup \{\beta\}$. If f is so extended, then $|f(\beta)| = 1$.*
(b) *If β_1 and β_2 are two distinct simple boundary points of Ω and f is extended to $\Omega \cup \{\beta_1, \beta_2\}$ as in (a), then $f(\beta_1) \neq f(\beta_2)$.*

Proof. Let $g\colon \Delta \to \Omega$ be the map inverse to f. Since Ω is bounded g is bounded.

Suppose f cannot be extended to $\Omega \cup \{\beta\}$. Then there exists a sequence $\{z_n\}$ in Ω converging to β, two distinct points w_1 and w_2 in $\overline{\Delta}$, such that $f(z_{2n}) \to w_1$ and $f(z_{2n+1}) \to w_2$ as $n \to \infty$.

Since β is a simple boundary point, we can find a curve $\gamma\colon [0,1] \to \Omega$, and a sequence $\{t_n\}$, $0 < t_1 < \cdots < t_n < \dots$ such that $\gamma(t) \in \Omega$ for $0 \leq t < 1$, $\gamma(1) = \beta$, and such that $\gamma(t_n) = z_n$.

Let $\Gamma(t) = f(\gamma(t))$ for $t \in [0,1)$. Then Γ is continuous on the half-open interval $[0,1)$. For $0 < r < 1$ let $K_r = g(\overline{B}(0,r)$. Then K_r is compact in Ω. Now, $\lim_{t\to 1} \gamma(t) = \beta$ and hence for a fixed r, there exists t^* (depending upon r), $0 < t^* < 1$, such that $\gamma(t) \notin K_r$ for $t^* < t < 1$. This means $|\Gamma(t)| = |f(\gamma(t))| > r$ for $t^* < t < 1$. It follows that $\lim_{t\to 1} |\Gamma(t)| = 1$. In particular $|w_1| = |w_2| = 1$, i.e., $w_k \in \mathbf{T}$ for $k = 1, 2$.

Since w_1 and w_2 are distinct points of $\mathbf{T}$, they define two complementary arcs in $\mathbf{T}$, such that the union of these complementary arcs is $\mathbf{T} \smallsetminus \{w_1, w_2\}$. Let $E = \Gamma([0,1))$, i.e., E is the image of Γ. We claim that at least one of the two complementary arcs, call it J, has the property that every radius ending at a point of J intersects E in a set with a limit point in $\mathbf{T}$, namely the end point of the radius.

Admitting the claim, part (a) is proved as follows. For every $\zeta \in J$, we have a sequence of points $t_n^* = t_n^*(\zeta) \in [0,1)$ with $t_n^* \uparrow 1$ such that $\Gamma(t_n^*) \in [0, \zeta)$ and $\lim_{n\to\infty} \Gamma(t_n^*) = \zeta$. It follows that $g(\Gamma(t_n^*)) = \gamma(t_n^*) \to \beta$ as $n \to \infty$. In particular if g has a radial limit for ζ, then the radial limit has to be β. Now g has a radial limit for almost every ζ in J (in fact in $\mathbf{T}$) by Fatou's theorem for bounded holomorphic functions (see Thm. 1.1.2 of Lecture 23). It follows that the radial limit function φ of g satisfies $\varphi|_J = \beta$ a.e. J. By Theorem 1.1.4 of Lecture 23, we have $g \equiv \beta$ on Δ. However g is univalent. This gives a contradiction, proving (a), provided we admit the claim we made.

Here is the proof of the claim we made. Suppose I and J are the two complementary arcs of $\mathbf{T}$ defined by w_1 and w_2, and suppose neither arc satisfies the claim made. Then there exists r_0, with $0 < r_0 < 1$, and points $\xi_1 \in I$, $\xi_2 \in J$ such that

$$(*) \qquad E \cap (r_0\xi_i, \xi_i) = \emptyset \qquad \text{for } i = 1, 2.$$

Consider the open subset U of Δ obtained by deleting the two radii $[0,\, \xi_1)$ and $[0,\, \xi_2)$ from Δ. Then U breaks up into two disjoint open sectors, U_1 containing w_1, and U_2 containing w_2. Since $\lim_{t\to 1} |\Gamma(t)| = 1$ and since $f(z_{2n}) \to w_1$, $f(z_{2n+1}) \to w_2$, there exists $N \in \mathbb{N}$ such that for $t \geq t_N$, $|\Gamma(t)| > r_0$ and for $n \geq N$, $f(z_{2n}) \in U_1$, $f(z_{2n+1}) \in U_2$.

Since $\Gamma(t)$ has absolute value greater than r_0 for $t \geq t_N$, therefore for $n \geq N$ $\Gamma([t_{2n}, t_{2n+1}])$ is a connected subset of $\{z \mid r_0 < |z| < 1\}$. On the other hand, by $(*)$, $\Gamma([t_{2n}, t_{2n+1}]) \cap [r_0\xi_i, \xi_i) = \emptyset$ for $i = 1, 2$ and hence

$$\Gamma([t_{2n}, t_{2n+1}]) \subset U = U_1 \sqcup U_2.$$

Since $\Gamma(t_{2n}) \in U_1$ and $\Gamma(t_{2n+1}) \in U_2$, the sets $U_1 \cap \Gamma([t_{2n}, t_{2n+1}])$ and $U_1 \cap \Gamma([t_{2n}, t_{2n+1}])$ give a disconnection of $\Gamma([t_{2n}, t_{2n+1}])$. This is a contradiction. This proves the claim and hence completes the proof of (a).

As for (b), suppose f has been extended to $\Omega \cup \{\beta_1, \beta_2\}$ as in (a). Let γ_1 and γ_2 be curves with parameter interval $[0, 1]$ such that $\gamma_i(t) \in \Omega$ for $t < 1$ and $\gamma_i(1) = \beta_i$, for $i = 1, 2$. Let $\Gamma_i = g \circ \gamma_i$. Clearly $\lim_{t \to 1} g(\Gamma_i(t)) = \lim_{t \to 1} \gamma_i(t) = \beta_i$. By Lindelöf's theorem (Theorem 1.2.1 of Lecture 23), it follows that the radial limit of g at $f(\beta_i)$ is β_i. Thus if $f(\beta_1) = f(\beta_2)$, then $\beta_1 = \beta_2$. $\square$

The following corollary is important enough to have the status of a theorem

Theorem 1.1.4 (Carathéodary)**.** *Let Ω be a bounded simply connected all of whose boundary points a simple. Then every univalent surjective map $f\colon \Omega \to \Delta$ extends uniquely to a continuous map $\overline{\Omega} \to \overline{\Delta}$. This extension is one-to-one, onto, and sends points on the boundary of Ω to points in $\mathbf{T}$.*

References

[R] W. Rudin, *Real and Complex Analysis*, (Third Edition), McGraw-Hill, New York, 1987.