LECTURE 23

Date of Lecture: April 3, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

The unit circle will be denoted $\mathbf{T}$ instead of C. As usual Δ will denote the open unit disc. As before, for $1 \le p \le \infty$ we identify $L^p([-\pi, \pi])$ with $L^p(\mathbf{T})$, where the σ-algebras in play are the Borel σ-algebras and the measures, the normalised Lebesgue on $[-\pi, \pi]$ (the usual Lebesgue measure divided by 2π) and the Haar measure on $\mathbf{T}$. The identification is the standard one.

1. Radial Limits, Fatou's Theorem

1.1. **Radial limits.** Let $u\colon \Delta \to \mathbf{C}$ be a measurable function, and suppose $\theta \in [-\pi, \pi]$ is such that $\lim_{r\to 1^-} u(re^{i\theta})$ exists. Then we say that the *radial limit* of u exists at θ, and the radial limit is $\lim_{r\to 1^-} u(re^{i\theta})$. As is common in these matters, we often say in this case that the radial limit of u exists for $e^{i\theta}$. If the radial limit exists for almost all $\zeta \in \mathbf{T}$ (with respect to the Haar measure) then the *radial limit* of u is any function φ on $\mathbf{T}$ which agrees with $e^{i\theta} \mapsto \lim_{r\to 1^-} u(re^{i\theta})$ for almost all θ. In somewhat greater detail, for u with radial limits almost everywhere on $\mathbf{T}$, suppose E is the set of Haar measure one given by

$$E = \{e^{i\theta} \mid \lim_{r\to 1^-} u(re^{i\theta}) \text{ exists}\}.$$

Consider the measurable function $R_u\colon \mathbf{T} \to \mathbf{C}$ such that

$$R_u(e^{i\theta}) = \begin{cases} \lim_{r\to 1^-} u(re^{i\theta}), & \theta \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.1.1. Any measurable function $\varphi\colon \mathbf{T} \to \mathbf{C}$ such that $\varphi = R_u$ a.e. on $\mathbf{T}$ is called a *radial limit of* u.

We often abuse terminology and speak of "the" radial limit of u in the above situation. We also call the equivalence class (modulo a.e.) of R_u the radial limit.

In Problem 3 of HW 9, you proved (more than) the following.

Theorem 1.1.2 (Fatou's theorem for bounded holomorphic functions)**.** *Let $f(z)$ be a bounded holomorphic function on Δ. Then*

(a) *The radial limit $\lim_{r\to 1^-} f(re^{i\theta})$ exists for almost all $\theta \in [-\pi, \pi]$.*
(b) *Let φ be the radial limit of f. Then $\varphi \in L^2(\mathbf{T})$ and its Fourier series is $\sum_{n=0}^{\infty} a_n e^{int}$, where $\sum_{n=0}^{\infty} a_n z^n$ is the power series expansion of f in Δ.*
(c) *$f = P[\varphi]$ (φ as in (b)).*

There is an obvious corollary, namely,

Corollary 1.1.3. *With above hypotheses and notations, $\varphi \in L^\infty(\mathbf{T})$ and $\|\varphi\|_\infty = \sup_{z\in\Delta} |f(z)|$.*

Proof. Let $M = \sup_{z\in\Delta} |f(z)|$. Since φ is the radial limit of f it is obvious $\|\varphi\|_\infty \leq M$. This means $\varphi \in L^\infty(\mathbf{T})$. On the other hand, $|f| = |P[\varphi]| \leq P[\|\varphi\|_\infty] = \|\varphi\|_\infty$, whence $M \leq \|\varphi\|_\infty$. □

Here is an extremely useful result concerning radial limits.

Theorem 1.1.4. *Let J be an arc of positive length in $\mathbf{T}$ and suppose f is a bounded holomorphic function on Δ and β a constant such that the radial limit $\varphi = \beta$ almost everywhere on J. Then $f \equiv \beta$ on Δ.*

Proof. Without loss of generality we may assume $\beta = 0$. Let n be a positive integer such the the arc subtends an angle larger than $2\pi/n$. Let $\zeta_k = e^{2\pi k/n}$, $k = 0, \dots, n-1$. Define $f_k \colon \Delta \to \mathbf{C}$ by the formula $f_k(z) = f(\zeta_k z)$, $z \in \Delta$, $k = 0, \dots, n-1$. Let $\tilde{f} = \prod_{i=0}^{n-1} f_k$. Now each f_k, and therefore $\tilde{f}$, is holomorphic. Moreover $\tilde{f}$ is bounded on Δ. If $\tilde{\phi}$ is its radial limit, then clearly from our hypotheses, $\tilde{\phi} = 0$. Since $\tilde{f} = P[\tilde{\varphi}]$, it follows that f is identically zero. The ring of holomorphic functions on Δ is an integral domain (recall problem from the mid-term exam). Hence some f_k is identically zero. From the definition of f_k, this means f is identically zero. □

1.2. **Lindelöf's Theorem.** The theorem we now prove is an example of a technique called the *Phragmén-Lindelöf method*. It is a technique for finding bounds for holomorphic functions on unbounded regions based on known bounds in the boundary (the usual maximum principle only applies for bounded regions). A modification of the technique sometimes yields bounds even when bounds are known in parts of the boundary of a bounded region. The theorem we now state is due to Lindelöf, the first of a long line of great Finnish function theorists (the Nevanlinna brothers and Ahlfors are his successors).

Theorem 1.2.1 (Lindelöf's Theorem)**.** *Let $\Gamma \colon [0,1] \to \overline{\Delta}$ be a curve such that $\Gamma(t) \in \Delta$ for $0 \leq t < 1$, and $\Gamma(1) = 1$. Let g be a bounded holomorphic function on the Δ such that $\lim_{t\to 1} g(\Gamma(t))$ exists, say*

$$\lim_{t\to 1} g(\Gamma(t)) = L.$$

Then the radial limit of g at $\theta = 0$ exists and is L.

Proof. The following proof is from [R, Thm. 12.10, pp.259–260].

Since g is bounded, by dividing by a suitable constant we assume, without loss of generality, that $|g| < 1$. We also assume, by subtracting by L if necessary, that $L = 0$, and this too is at no cost to generality. Let $\varepsilon > 0$ be given. There exists $t_0 \in [0, 1)$ such that

$$|g(\Gamma(t))| < \varepsilon \qquad (t > t_0), \tag{1.2.1.1}$$

and such that, with $r_0 = \mathrm{Re}(\Gamma(t_0))$,

$$\mathrm{Re}(\Gamma(t)) > r_0 > \frac{1}{2} \qquad (t > t_0). \tag{1.2.1.2}$$

Pick r with $r_0 < r < 1$. Let $\Omega = B(0, 1) \cap B(2r, 1)$. Since $1 > r > r_0 > 1/2$, the left-most boundary point of the disc $B(2r, 1)$, namely $2r - 1$, is positive and less than 1. In particular Ω is non-empty. Clearly Ω is symmetric about the vertical line $\mathrm{Re}(z) = r$ and also about the real axis. This means that $z \in \Omega$ if and only if $2r - z \in \Omega$), and that $z \in \Omega$ if and only if $\bar{z} \in \Omega$.

On Ω define a holomorphic function $h(z)$ by the formula

$$h(z) = g(z)\overline{g(\bar{z})}g(2r - z)\overline{g(2r - \bar{z})}. \tag{1.2.1.3}$$

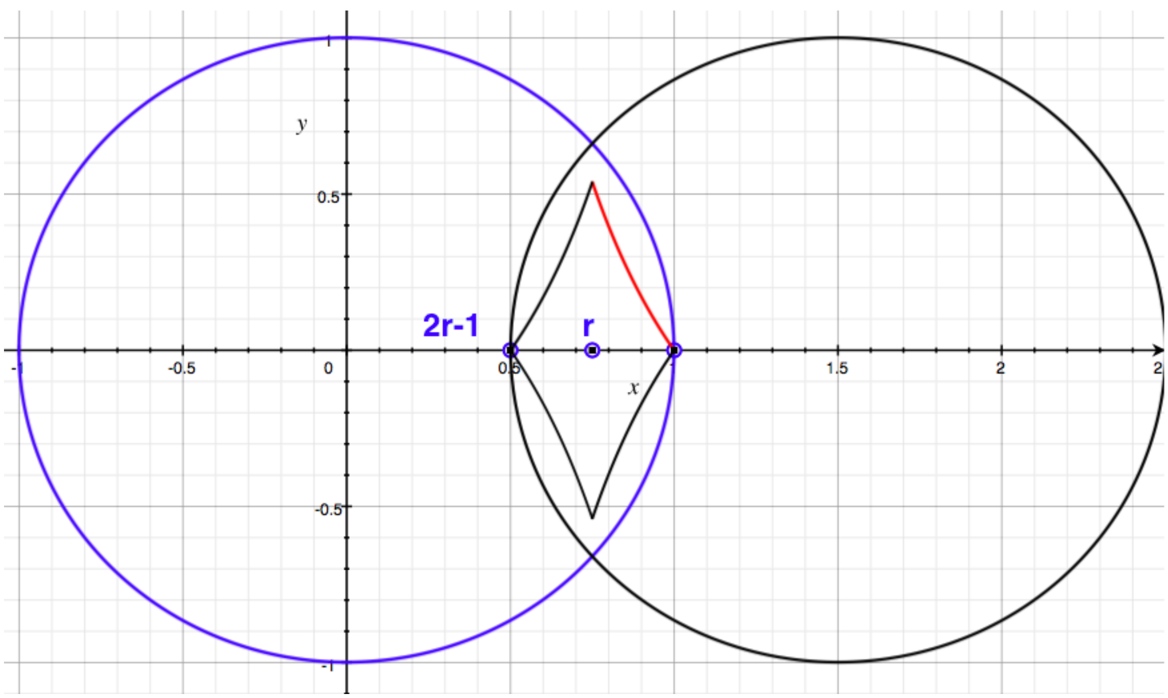
Note that

$$h(r) = |g(r)|^4. \tag{1.2.1.4}$$

We will show $h(r) \leq \varepsilon$. In view of (1.2.1.4) this will be enough.

Let $t_1 = \max\{t \in [t_0, 1] \mid \mathrm{Re}(\Gamma(t)) = r\}$. Clearly $t_0 < t_1 < 1$. Let $E_1 = \Gamma([t_1, 1])$, and let E_2 be its reflection in the real axis. Next reflect $E_1 \cup E_2$ along the other axis of symmetry of Ω, namely, the line $x = r$ to get E', and set $E = E_1 \cup E_2 \cup E'$. E is clearly symmetric about the two known axes of symmetry of Ω, namely the x-axis, and the vertical line $x = r$. We point out that the reflection of a point z about the line $x = r$ is $2r - z$. This means the function h is symmetric about $x = r$, as an easy inspection of the formula (1.2.1.3) shows.

Here is a picture for $r = 3/4$, and $\Gamma(t) = t + i(t^{-1.5} - 1)$ for $r \leq t \leq 1$ and equal to $t + i(r^{-1.5} - 1)$ for $0 \leq t \leq r$. Note that $t_1 = r = 3/4$. The red curve is E_1. The rest of the curves diamond are the various reflections of E_1 and the resulting curvilnear diamond shape is E. The centre of the interior of the diamond is r.



Coming back to the general case, note that the right end point of E_1, and hence of E, is $z = 1$, and therefore the left end point of E is the reflection of this point in the line $x = r$, i.e., it is the point $z = 2r - 1$. Neither end point lies in Ω. All other points of E do indeed lie in Ω.

Now $E \cap \Omega = E \setminus \{2r - 1, 1\}$. By (1.2.1.1) and the fact that $|g| < 1$, we clearly have

$$|h(z)| < \varepsilon \qquad (z \in E \setminus \{2r - 1, 1\}) \tag{1.2.1.5}$$

We want to conclude that $|h(r)|$ $(= h(r))$ is less then ε. If $\Gamma(t_1) = r$, we are done by (1.2.1.1)and (1.2.1.4). Otherwise, r is some sort of interior point of an open set bounded by E (we will make this precise in a moment). However, we do not have a bound for $|h|$ on E, but only on $E \setminus \{2r-1, 1\}$, and so the Maximum Principle does not apply. This is the sort of situation where the Phragmén-Lindelöf technique is often successful.

Since we have already taken care of the case where $\Gamma(t_1) = r$, let us assume, $\mathrm{Im}(\Gamma(t_1)) \neq 0$. Let K be the union of E and all bounded components of the complement of E in $\mathbf{C}$. Then K is compact, has E as its boundary, and r as an interior point. For $c > 0$ define $(1-z)^c$ and $(2r-1-z)^c$ on

$$U = \{z \mid 2r-1 < \mathrm{Re}(z) < 1\}$$

the following way. Note that the strip U is simply connected and $1-z$ and $2r-1-z$ are holomorphic on U and nowhere vanishing there. If $f(z)$ is such a nowhere vanishing function on U, pick a branch of $\log(f(z))$ on U and define $f(z)^c$ as $\exp(c \cdot \log f(z))$. Now that $(1-z)^c$ and $(2r-1-z)^c$ are defined on U, define, for $c > 0$, $h_c \colon K \to \mathbf{C}$ as follows.

$$h_c(z) = \begin{cases} h(z)(1-z)^c(2r-1-z)^c, & \text{if } z \in K \setminus \{2r-1, 1\}, \\ 0, & \text{if } z \in \{2r-1, 1\}. \end{cases}$$

Now h_c is continuous on K, holomorphic in the interior of K, and on the boundary E of K, $|h_c| < \varepsilon$, by (1.2.1.5). Since r is an interior point of K, by the Maximum Principle, $|h_c(r)| < \varepsilon$. Letting $c \to 0$ we obtain $h(r) \leq \varepsilon$. □

References

[R] W. Rudin, *Real and Complex Analysis*, (Third Edition), McGraw-Hill, New York, 1987.