LECTURES 21 AND 22

Date of Lectures: March 27 and 29, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

We will change notation from this lecture. The unit circle will be denoted $\mathbf{T}$ instead of C. As usual Δ will denote the open unit disc. As before, for $1 \le p \le \infty$ we identify $L^p([-\pi,\, \pi])$ with $L^p(\mathbf{T})$, where the σ-algebras in play are the Borel σ-algebras and the measures, the normalised Lebesgue on $[-\pi,\, \pi]$ (the usual Lebesgue measure divided by 2π) and the Haar measure on $\mathbf{T}$. The identification is the standard one.

1. Some real analysis

Fix a closed interval $[a,\, b]$ for the discussion that follows.

1.1. Functions of bounded variation.

Functions of bounded variation α are the functions for which the traditional Riemann-Steiljes $\int_a^b f(x)d\alpha(x)$ is defined. They were defined by Jordan in 1881 to work out convergence of Fourier series. Later this was generalised to the Lebesgue-Steiljes integral. Functions of bounded variation are precisely those that can be written as the difference of two non-decreasing functions. Right continuous (or left continuous) functions of bounded variation α with $\alpha(a) = 0$ can be identified with signed Borel measures on $[a,\, b]$. Indeed, given a right continuous bounded variation function α, one defines the measure of $(p,\, q] \subset [a,\, b]$ to be $\alpha(p) - \alpha(q)$ and the measure of $\{a\}$ to be $\alpha(a)$. This defines a signed measure μ on the sigma-algebra generated by such sets, namely the Borel sigma algebra. Conversely, given a signed measure μ on $[a,\, b]$ we define α by $\alpha(a) = 0$ and $\alpha(x) = \mu([a,\, x])$.

See [J], [R], and [T] for more details.

One of the first things we need to know is the following.

Theorem 1.1.1 (Lebesgue)**.** *Let $f\colon [a,\, b] \to \mathbf{R}$ be a monotone function. Then f' exists almost everywhere.*

Proof. See [J, Chapter 2, Theroem 1.2] or [T, Theorem 2.1, p.167]. □

The definition of a function of bounded variation is the following.

Definition 1.1.2 (Function of bounded variation)**.** A function $f\colon [a,\, b] \to \mathbf{R}$ is said to be of *bounded variation* if

$$\sup\left\{\sum_{i=1}^{n}\Big|f(x_i) - f(x_{i-1})\Big|\right\} < \infty$$

where the supremum is taken over all partitions $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of $[a,\, b]$.

Theorem 1.1.3 (Jordan)**.** *A function $f\colon [a,\, b] \to \mathbf{R}$ is of bounded variation if and only if it can be written as the difference of two non-decreasing functions. In particular, if f is of bounded variation, then f' exists almost everywhere.*

Proof. See [T, p.31, Thm. 1.3] or [J, Chapter 2, Thm. 3.1]. □

Remark 1.1.4. If f is of bounded variation and is continuous at x_0 then f can be written as the difference of two monotone functions which are both continuous at x_0. See [J, Chapter 2, Thm. 3.2].

1.2. **Absolutely continuous functions.** The notion is due to Vitali, and predates the definition of absolutely continuous measures. One way of thinking of them is that they are the bounded variation functions associated with signed measures which are absolutely continuous with respect to the Lebesgue measure. Here is the definition

Definition 1.2.1 (Absolutely continuous functions)**.** Let $f\colon [a,\, b] \to \mathbf{R}$ be a function. It is said to be *absolutely continuous* if given $\epsilon > 0$, there exists a $\delta > 0$, such that for any finite collection of non-overlapping intervals $\{(a_i,\, b_i)\}$ with $\sum_i (b_i - a_i) < \delta$, we have

$$\sum_i \Big| f(b_i) - f(a_i) \Big| < \epsilon.$$

Theorem 1.2.2. *If f is absolutely continuous on $[a,\, b]$ then it is of bounded variation, and hence f' exists almost everywhere on $[a, b]$.*

Proof. See [J, Chap. 3,Thm. 2.2]. □

Theorem 1.2.3. *A function $f\colon [a,\, b] \to \mathbf{R}$ is absolutely continuous if and only if there exists $\varphi \in L^1([a,\, b])$ such that*

$$f(x) = f(a) + \int_a^x \varphi(t)dt.$$

In this case $f' = \varphi$ a.e. on $[a,\, b]$.

Proof. See [J, Chap. 3, Thms. 2.2 and 2.3]. See also [T, p. 177, Thm. 3.6] as well as [R, p.148, Thm. 7.18]. □

1.3. **Integration by parts.** Suppose f and g are absolutely continuous on $[a,\, b]$. We would like integration by parts to hold for fg'. The product rule gives $fg' = (fg)' - gf'$ almost everywhere. The problem is that it is not *a priori* clear that $\int_a^x (fg)'(t)dt = f(x)g(x) - f(a)g(a)$. This would be true if fg is also absolutely continuous. This is proven in Lemma 1.3.3 below. But first we need to define an auxiliary function.

For our absolutely continuous function f, let $V(f)\colon [a,\, b] \to \mathbf{R}$ be the function given by

$$(V(f))(x) = |f(a)| + \int_a^x |f'(t)|dt. \tag{1.3.1}$$

By Theorem 1.2.3, $V(f)$ is absolutely continuous. Moreover

$$|f| \leq V(f). \tag{1.3.2}$$

Indeed, clearly $f \leq V(f)$ and $V(f) = V(-f)$, giving (1.3.2). Note also that $V(f)$ is non-decreasing. We are now in a position to prove the following:

Lemma 1.3.3. *Suppose f and g are absolutely continuous on $[a, b]$. Then the product fg is also absolutely continuous.*

Proof. Let $\epsilon > 0$ be given. We have $\delta > 0$ such that for any finite collection of non-overlapping intervals $\{(a_i, b_i)\}$ with $\sum_i (a_i - b_i) < \delta$ we have

$$\sum_i \Big| f(a_i) - f(b_i) \Big| < \frac{\epsilon}{V(f)(b) + V(g)(b)}$$

and

$$\sum_i \Big| g(a_i) - g(b_i) \Big| < \frac{\epsilon}{V(f)(b) + V(g)(b)}$$

For such a collection of non-overlapping intervals $\{(a_i, b_i)\}$ we have

$$\begin{aligned}\sum_i \Big| f(a_i)g(a_i) - f(b_i)g(b_i) \Big| &\le \sum_i \Big| f(a_i)g(a_i) - f(a_i)g(b_i) \Big| + \sum_i \Big| f(a_i)g(b_i) - f(b_i)g(b_i) \Big| \\ &= \sum_i \Big| f(a_i) \Big| \Big| (g(a_i) - g(b_i)) \Big| + \sum_i \Big| (f(a_i) - f(b_i)) \Big| \Big| g(b_i) \Big| \\ &\le V(f)(b) \sum_i \Big| (g(a_i) - g(b_i)) \Big| + V(g)(b) \sum_i \Big| (f(a_i) - f(b_i)) \Big| \\ &< \epsilon.\end{aligned}$$

□

Here is the theorem we want:

Proposition 1.3.4 (Integration by parts)**.** *Suppose f and g are absolutely continuous on $[a, b]$. Then*

$$\int_a^x f(t)g'(t)dt = f(x)g(x) - f(a)g(a) - \int_a^x g(t)f'(t)dt$$

for all $x \in [a\, b]$.

Proof. We have $fg' = (fg)' - gf'$. It follows that

$$\int_a^x f(t)g'(t)dt = \int_a^x (fg)'(t)dt - \int_a^x g(t)f'(t)dt.$$

By Lemma 1.3.3, fg is absolutely continuous, and hence by Theorem 1.2.3 we have,

$$\int_a^x (fg)'(t)dt = f(x)g(x) - f(a)g(a).$$

This gives the result. □

2. **Fatou's Theorem**

There are various versions of Fatou's Theorem (see HW 9, problem 3, for the version for bounded holomorphic functions on Δ–and this is the version that is most useful for proving Picard's theorem). The main theme here is the existence of radial limits. In greater detail, the question is this: Suppose u is harmonic on Δ, and $u = P[\varphi]$ for some φ on $\mathbf{T}$. Under what circumstances is $\varphi(e^{i\theta}) = \lim_{\to 1^-} u(re^{i\theta})$? Here is the version from which the version in HW 9 follows, after a little bit of work.

Theorem 2.1 (Fatou's Theorem for $L^1(\mathbf{T})$)**.** *Let $\varphi \in L^1(\mathbf{T})$, $F(x) = \int_{-\pi}^{x} \varphi(t)dt$, and $u = P[\varphi]$. If F is differentiable at θ_0, then*

$$\lim_{r \to 1^-} u(re^{i\theta_0}) = F'(\theta_0),$$

where the above equality means that the limit on the left side exists and equals the number on the right. In particular the radial limits $\lim_{r\to 1^-} u(re^{i\theta})$ exist for almost all θ.

Remark: Note that F is absolutely continuous and hence Theorem 1.2.2 and Theorem 1.2.3 apply to it. In particular, F' exists almost everywhere and $F' = \varphi$ almost everywhere. Fatou's Theorem for $L^1(\mathbf{T})$ then says that the radial limits converge to φ almost everywhere, giving an answer of sorts for the Dirichlet problem on the disc Δ when the boundary function is L^1. The following proof, which avoids approximate identities, is via a private communication from Elias Katsoulis.

Proof. Without loss of generality assume $\theta_0 = 0$. In the computations below, in the first line we are using the fact that $P_r(-t) = P_r(t)$ for all $t \in \mathbf{R}$, and in the second line we are using Proposition 1.3.4. We have,

$$\begin{aligned} u(r) - F'(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\varphi(t) - F'(0))P_r(t)dt \\ &= \frac{1}{2\pi} \left[\Big(F(t) - tF'(0)\Big) P(t) \right]_{t=-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} (F(t) - tF'(0))P_r'(t)dt \\ &= \frac{1}{2\pi} \frac{1-r^2}{(1+r)^2} (F(\pi) - F(-\pi) - 2\pi F'(0)) - \frac{1}{2\pi} \int_{-\pi}^{\pi} (F(t) - tF'(0))P_r'(t)dt. \end{aligned}$$

The first term above converges to 0 as $r \to 1^-$. We have to show the same for the rest. For δ, $0 < \delta < \pi$, we have,

$$(*) \qquad \int_{-\pi}^{\pi} (F(t) - tF'(0))P_r'(t)dt = \int_{-\delta}^{\delta} (F(t) - tF'(0))P_r'(t)dt + \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right).$$

If $|t| > \delta$ we have

$$\left| P_r'(t) \right| = \left| \frac{(1-r^2)(-2r\sin t)}{(1 - 2r\cos t + r^2)^2} \right| \le \frac{2r(1-r^2)}{1 - 2r\cos\delta + r^2}.$$

Moreover, F being continuous, $|F(t) - tF'(0)|$ is bounded on $[-\pi, \pi]$. Thus the two integrals within parenthesis in $(*)$ clearly $\to 0$ and $r \to 1^-$, *whatever be δ in the range* $0 < \delta < \pi$. We have to choose $\delta > 0$ such that the integral

$$\int_{-\delta}^{\delta} (F(t) - tF'(0))P_r'(t)dt$$

is small. An easy change of variables, using the fact that $P_r'(-t) = -P_r'(t)$ for all t, shows that

$$\int_{-\delta}^{0} (F(t) - tF'(0))P_r'(t)dt = -\int_{0}^{\delta} (F(-t) + tF'(0))P_r'(t)dt.$$

Thus

$$(**) \qquad \int_{-\delta}^{\delta} (F(t) - tF'(0))P_r'(t)dt = \int_{0}^{\delta} \left(\frac{F(t) - F(-t)}{2t} - F'(0) \right) 2tP_r'(t)dt$$

Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$(\dagger) \qquad \left| \frac{F(t) - F(-t)}{2t} - F'(0) \right| < \epsilon,$$

whenever $|t| < \delta$. Then, from $(**)$ and $(\dagger)$ we have:

$$\begin{aligned}
\left| \int_{-\delta}^{\delta} (F(t) - tF'(0)) P_r'(t) dt \right| &< \epsilon \int_0^{\delta} |2tP_r'(t)| dt \\
&= \epsilon \int_0^{\delta} 2t(-P_r'(t)) dt \\
&\leq \epsilon \int_0^{\pi} 2t(-P_r'(t)) dt \\
&= 2\epsilon \left(\Big[t(-P_r(t)) \Big]_{t=0}^{r} + \int_0^{\pi} P_r(t) dt \right) \\
&= 2\pi\epsilon \left[-\frac{1 - r^2}{(1 + r)^2} + 1 \right] \\
&\leq 2\pi\epsilon
\end{aligned}$$

since $1 - \frac{1-r^2}{(1+r)^2} \leq 1$ for $0 \leq r < 1$. (Note that since $P_r(t)$ is an even function, we have $\int_0^{\pi} P_r(t) dt = (1/2) \int_{-\pi}^{\pi} P_r(t) dt = (1/2)(2\pi) = \pi$. This explains the last but one step in the chain of computations above.) □

3. **Fourier Series**

Recall that the Hilberts spaces $L^2(\mathbf{T})$ and $\ell^2(\mathbb{Z})$ are isometrically isomorphic, with the isomorphism $\Phi \colon L^2(\mathbf{T}) \xrightarrow{\sim} \ell^2(\mathbb{Z})$ being $\varphi \mapsto \{\widehat{\varphi}(n)\}$. Let $\varphi \in L^2(\mathbf{T})$. Since the Haar measure on $\mathbf{T}$ is a finite positive measure on $\mathbf{T}$, $P[\varphi]$ is a harmonic function on u. If φ is $\mathbf{R}$-valued, so is u, then with $\sigma \colon \mathbf{C} \to \mathbf{C}$ being the conjugation map $z \mapsto \bar{z}$, we have

$$(*) \qquad \begin{aligned}
\widehat{\varphi}(-n) &= \frac{1}{2\pi} \int_{\pi}^{\pi} e^{int} \varphi(t) dt \\
&= \frac{1}{2\pi} \sigma \left(\int_{\pi}^{\pi} e^{-int} \varphi(t) dt \right) \\
&= \overline{\widehat{\varphi}(n)}.
\end{aligned}$$

3.1. **The Fourier series of P_r.** Suppose $n \geq 0$. Let $f(z) = z^n$. For $0 \leq r < 1$ we have

$$\begin{aligned} r^n e^{in\theta} &= f(re^{in\theta}) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta - t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i(\theta)-t})P_r(t)dt \\ &= e^{in\theta}\frac{1}{2\pi}\int_{-\pi}^{\pi} r^{-int}P_r(t)dt \\ &= \widehat{P}_r(n)e^{in\theta}. \end{aligned}$$

This means

$$\widehat{P}_r(n) = r^n, \qquad (n \geq 0).$$

Applying $(*)$ to the above (since P_r is real valued), we get

$$\widehat{P}_r(n) = r^{|n|}. \tag{3.1.1}$$

3.2. **Fourier coefficients of Poisson transforms.** As usual, if f is a function on Δ, then for r, $0 \leq r < 1$, we define $f_r \colon \mathbf{T} \to \mathbf{C}$ by the formula

$$f_r(e^{it}) := f(re^{it}).$$

The main result is the following.

Lemma 3.2.1. *Suppose $\varphi \in L^2(\mathbf{T})$ and u the harmonic function on Δ given by $u = P[\varphi]$. Then*

$$\widehat{u}_r(n) = r^{|n|}\widehat{\varphi}(n),$$

for $n \in \mathbb{Z}$ and $0 \leq r < 1$.

Proof. We have, for n and r in the required range,

$$\begin{aligned} \widehat{u}_r(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\varphi(\theta - t)dt\Big)d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}P_t(t)\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in(\theta-t)}\varphi(\theta - t)d\theta\Big)dt \\ &= \widehat{P}_r(n)\widehat{\varphi}(n). \end{aligned}$$

Using (3.1.1) we are done. □

References

[J] R-Q. Jia, *Math 418 Notes*, University of Alberta, `https://sites.ualberta.ca/∼rjia/Math418/Notes/`.

[R] W. Rudin, *Real and Complex Analysis*, (Third Edition), McGraw-Hill, New York, 1987.

[T] A. Torchinsky, *Real Variables*, Addison-Wesley, 1988.