LECTURE 20

Date of Lecture: March 22, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

1. The Poisson kernel on the unit disc

Let Δ be the unit disc and C its bounding circle. We use the following standard conventions. A (Borel) measurable function φ on C will be identified with the corresponding function on $[-\pi, \pi]$. In other words, if $\theta \in [-\pi, \pi]$, we will feel free to write $\varphi(e^{i\theta})$ or $\varphi(\theta)$ for the same number. We denote by $L^p(C)$ the space $L^p(C, \sigma)$ where σ is the measure given by the uniform probability measure on C, i.e., the arc-length measure divided by 2π. We refer to σ as the *Haar measure* on C.[1] For $1 \leq p \leq \infty$, we identify $L^p[-\pi, \pi]$ with $L^p(C)$. A property is true *almost everywhere* on C, if is true almost everywhere σ. Finally, as is standard, the abbreviation a.e. will stand for almost everywhere. Note that these notions (i.e., L^p or a.e.) on an interval in $\mathbf{R}$ are by default with respect to the Lebesgue measure.

1.1. Poisson kernel.

Let r be a real number, $0 \leq r < 1$. For $\theta \in [-\pi, \pi]$ define

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \tag{1.1.1}$$

Set $a = re^{i\theta}$ and $z = e^{it}$. An easy calculation shows,

$$P_r(\theta - t) = \frac{1 - |a|^2}{|z - a|^2} = \mathrm{Re}\left[\frac{z + a}{z - a}\right]. \tag{1.1.2}$$

Therefore, via Problem 1 of HW 6, if u is continuous on $\overline{\Delta}$ and harmonic in Δ, we have

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})P_r(\theta - t)dt \tag{1.1.3}$$

for $0 \leq r < 1$ and $\theta \in [-\pi, \pi]$. The family of functions $\{P_r(\theta)\}$ is called the *Poisson kernel* on C or on Δ. By obvious translation and scaling, it is clear one has Poisson kernels for all discs/circles of finite radius in $\mathbf{C}$, but we will concentrate on Δ.

[1] Usually, when one treats matters from the point of view of this lecture, the unit circle C is denote $\mathbf{T}$. We do not wish to switch notations in the middle of the course, so we will continue to use the symbol C, with great reluctance.

1.2. **The Poisson integral.** Now suppose $f \in L^1[-\pi, \pi]$ (f real-valued). Recall, we also regard f as a member of $L^1(C)$ (see the discussion at the beginning of this section). One defines the *Poisson transformation* or the *Poisson integral* of f to be the function $P[f]$ given by

$$P[f](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P_r(\theta - t)dt \tag{1.2.1}$$

for r, θ such that $0 \le r < 1$ and $-\pi \le \theta \le \pi$.[2] Applying (1.1.2) we get

$$\begin{aligned} P[f](z) &= \mathrm{Re}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} f(t)dt\right] \\ &= \frac{1}{2\pi}\mathrm{Re}\int_{-\pi}^{\pi}\left[\frac{2e^{it}}{e^{it}-z} - 1\right] f(t)dt \\ &= -1 + \frac{1}{2\pi}\mathrm{Re}\int_{-\pi}^{\pi} \frac{2e^{it}}{e^{it}-z} f(t)dt. \end{aligned} \tag{1.2.2}$$

The integral on the last line of (1.2.2) can be written in the form

$$\int_X \frac{1}{\varphi(t) - z} d\mu(t)$$

where $X = [-\pi, \pi]$, $\varphi(t) = e^{it}$, and μ is the Borel measure on $[-\pi, \pi]$ given by $E \mapsto \int_E 2e^{it} f(t)dt$. From Theorem 1.1 of Lecture 2, it follows that the above integral defines a holomorphic function on $\mathbf{C} \setminus C$ (in particular on Δ). In fact if K is a closed subset of C such that f vanishes outside K, then the same argument[3] shows that the integral defines a holomorphic function on $\mathbf{C} \setminus K$, and therefore can be extended to the part of the circle C outside K. In particular $P[f]$ is harmonic, being the real part of a holomorphic function. These statements are included in the following proposition which proves a little more.

Proposition 1.2.3. *Let $f \in L^1[-\pi, \pi]$ and $P[f]\colon \Delta \to \mathbf{R}$ be the function defined by the Poisson integral* (1.2.1)*. Then*

(a) *P_f is harmonic.*
(b) *If K is a closed subset of C such that f is zero on $C \setminus K$, then P_f extends as a harmonic function to $\mathbf{C} \setminus K$.*
(c) *If g is another member of $L^1[-\pi, \pi]$, and α and β are real numbers, then $P[\alpha f + \beta g] = \alpha P[f] + \beta P[g]$.*
(d) *If g is another member of $L^1[-\pi, \pi]$ and $f \ge g$ a.e., then $P[f] \ge P[g]$ on Δ.*
(e) *If f is a constant e.e., say $f = c$ a.e. on C, where c is a real number, then $P_f \equiv c$ on Δ.*

Observation: If $f = g$ a.e. on C then clearly g is also in $L^1[-\pi, \pi]$ and $P[f] = P[g]$. We use this property without comment in what follows.

Proof. Parts (a) and (b) have been proven in the discussion above the statement of the proposition. Part (c) is obvious. In view of (c), to prove (d) it is enough to prove that if $f > 0$ then $P[f] > 0$. But this is clear from (1.1.1) and (1.2.1),

[2]The formula shows that $P[f]$ is the *convolution* of f and P_r, i.e. $P[f] = f * P_r$.
[3]Replace the space $X = [-\pi, \pi]$ by $X = \{t \in [-\pi, \pi] \mid e^{it} \in K\}$.

for $P_r(t-\theta) > 0$ when $0 \le r < 1$. Finally, if one applies (1.1.3) to the harmonic function $u \equiv c$, we get (e). $\square$

Remark 1.2.4. In parts (d) and (e) of the proposition, please note that the statement is only being made for Δ. This is for two reasons. First, the inequality $P_r(t-\theta) > 0$ no longer holds when $r > 1$ and next, (1.1.3) is only true for $re^{i\theta} \in \Delta$.

1.3. **The Dirichlet problem on $\overline{\Delta}$.** The problem, attributed to Dirichlet by Riemann, is the following (in its modern form). Suppose Ω is a bounded region in $\mathbf{C}$ and boundary $\partial\Omega$ is nice in some sense. Say, for simplicity that Ω is simply connected $\partial\Omega$ is given by a closed Jordan path γ (i.e., $\mathbf{C} \setminus \gamma^*$ has only two components, one of which is Ω. Suppose f function on $\partial\Omega$. Can one extend f to a function u on $\overline{\Omega}$ such that u is harmonic on Ω and equals f on the boundary of Ω and in some sense, for a boundary points p of Ω, $f(p)$ is the limit of $u(z)$ as z approaches the p from Ω. The sense of limit could be L^p convergence, pointwise convergence, weak-* convergence $\ldots -$ in a suitable function space.

The following theorem is a solution the Dirichlet problem on $\overline{\Delta}$.

Theorem 1.3.1. *Let f be a real-valued Borel measurable function on $[-\pi, \pi]$ which is integrable with respect the Lebesgue measure, and as usual regard f also as a function on C. Let $e^{i\theta_0}$ be a point of continuity for $f(e^{it})$. Then*

$$\lim_{\substack{z \to e^{i\theta_0} \\ z \in \Delta}} P[f](z) = f(e^{i\theta_0}).$$

In particular if f is continuous on C, then P_f extends to a continuous function on $\overline{\Delta}$ which agrees with f on C.

Proof. By replacing f by $f - f(\theta_0)$ if necessary, and using parts (c) and (e) of Proposition 1.2.3, we will assume, without loss of generality, that $f(\theta_0) = 0$.

Let $\epsilon > 0$ be given. Since f is continuous and vanishes at $e^{i\theta_0}$, we can find a closed arc J_1 in C containing $e^{i\theta_0}$ as an interior point, such that $|f| < \epsilon/2$ on J_1. Let J_2 be the closed arc in C having the same boundary points as J_1 and such that $J_1 \cup J_2 = C$ (the so-called complementary arc). Let $f_1 = f\chi_{J_1}$ and $f_2 = f\chi_{J_2}$. Now $|f_1| < \epsilon/2$ whence by parts (d) and (e) of Proposition 1.2.3 we have

$$(*) \qquad \left| P_{f_1}(z) \right| \le \frac{\epsilon}{2} \qquad (z \in \Delta)$$

Since f_2 vanishes outside J_2, therefore by part (b) of Proposition 1.2.3 we have $P[f_2]$ extends as a harmonic function (in particular as a continuous function) on $\mathbf{C} \setminus J_2$. Now $e^{i\theta_0} \in \mathbf{C} \setminus J_2$. Moroever for $\theta \in C \setminus J_2$ and $t \in J_2$ we have

$$f_2(t)P_r(t-\theta) = f_2(t)\frac{1 - |e^{i\theta}|^2}{|e^{it} - e^{i\theta}|^2} = 0.$$

Since f_2 vanishes outside J_2, this means

$$P[f_2](e^{i\theta}) = 0 \qquad (e^{i\theta} \notin J_2).$$

In particular, since $e^{i\theta_0} \in \mathbf{C} \setminus J_2$ so $P[f_2]$ is continuous at $e^{i\theta_0}$ and vanishes there, we can find $\rho > 0$ such that

$$(**) \qquad \left| P[f_2](z) \right| < \frac{\epsilon}{2} \qquad (|z - e^{i\theta_0}| < \rho).$$

Since $f = f_1 + f_2$ a.e. on C (they fail to be equal only on the two common boundary points of J_1 and J_2), we have $P[f_1 + f_2] = P[f]$ by part (c) of Proposition 1.2.3. This along with $(*)$ and $(**)$ gives us

$$\Big|P[f](z)\Big| \leq \Big|P[f_1](c)\Big| + \Big|P[f_2](z)\Big| < \epsilon$$

for $z \in \Delta \cap \{z \mid |z - e^{i\theta_0}| < \rho\}$. This is what we were required to show. □

2. **Local Averaging Property**

In this section will now show that if a continuous has the local averaging property in a region, then it must be harmonic.

2.1. **The maximum principle.** Here is one way of stating the maximum principle for functions with the local averaging property.

Lemma 2.1.1. *A non-constant continuous real-valued function u on a region Ω with the local averaging property has neither a maximum nor a minimum in Ω. In particular, if Ω is a bounded set and u extends to a continuous function on $\overline{\Omega}$, then u attains its maximum and minimum on the boundary of Ω.*

Proof. If $a \in \Omega$ is a point such that $u(z) \leq u(a)$ for all $z \in \Omega$, then we have $u - u(a) \geq 0$ on Ω. If $\overline{B}(a, \rho)$ is a closed disc in Ω with centre a, such that u has the averaging property on $\overline{B}(a, r)$ for all $r \leq \rho$, then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(u(a + re^{it}) - u(a))dt = 0$$

for all $0 < r \leq \rho$. Since $u - u(a)$ is continuous and non-negative, this means $u(z) = u(a)$ for z such that $|z - a| = r$ and $r \leq \rho$. In other words u is constant on $\overline{B}(a, \rho)$. Thus the set $\{z \in \Omega \mid u(z) = u(a)\}$ is open and closed in Ω, and since Ω is connected, this means u is constant. □

2.2. **Harmonicity.** Let us (temporarily) agree to say that a function is harmonic on a closed disc if it is continuous on the closed disc and harmonic in the interior of the dic. Lemma 2.1.1 more or less proves that continuous functions with the local averaging are harmonic. A useful observation is that translations and dilations do not change either the averaging property or harmonicity. In other words if $\varphi \colon \overline{B}(a, r) \to \overline{\Delta}$ is the obvious map $z \mapsto (1/r)(z - a)$ then u is harmonic on $\overline{B}(a, r)$ (resp. the average of u over circles of radius $\leq r$ centred at a is $u(a)$) if and only if $u \circ \varphi^{-1}$ is harmonic (resp. the average of $u \circ \varphi^{-1}$ over circles centred at 0 of radius ≤ 1 is $u \circ \varphi^{-1}(a)$) on $\overline{\Delta}$. Note also that the local averaging property is true for u if and only if $u \circ \varphi^{-1}$ has the local averaging property.

Theorem 2.2.1. *Suppose u is a continuous real-valued function with the local averaging property on a region Ω. Then u is harmonic.*

Proof. Let $a \in \Omega$ and $\rho > 0$ such that $\overline{B}(a, \rho) \subset \Omega$ and

$$u(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a + re^{it})dt$$

for $r \leq \rho$. From the comments made before the statement of the theorem, we may assume that $a = 0$ and $\rho = 1$, i.e., $\overline{B}(a, \rho) = \overline{\Delta}$. Let $f = u|_C$. Let $v = P[f]$. Then u and v have the local averaging property on Δ and are continuous on $\overline{\Delta}$.

Therefore $u - v$ has the these properties. Moreover $(u - v)|_C = 0$. It follows from Lemma 2.1.1 that $u = v$ on Δ. Since $v = P[f]$ it is harmonic on Δ, and hence so is u. $\square$