LECTURE 2

Date of Lecture: January 7, 2017

As usual, this is only a summary, and not all proofs given in class are here. Theorem 1.1 is a more complete statement than what I put up in class (in class I did not mention the integral formula for the coefficients of the power series expansion), and so I have given a proof. As for Proposition 3.1.1, the proof in class was hurried, and hence I decided to write out one here.

1. **Power Series**

Recall that by definition, a complex measure on a measurable space has finite total variation, whence every bounded measurable **C**-valued function is integrable with respect to that measure.

Theorem 1.1. *Let $(X, \mathscr{F}, \mu)$ be a complex measure space, $\varphi\colon X \to \mathbf{C}$ a measurable function, and Ω a region in $\mathbf{C}$ such that $\varphi(X) \cap \Omega = \emptyset$. Let*

$$f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z} \qquad (z \in \Omega). \tag{1.1.1}$$

Then $f(z)$ is holomorphic on Ω and for any disc $B(a,r)$ contained in Ω, $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n. \tag{1.1.2}$$

The coefficients c_n satisfy

$$c_n = \int_X \frac{d\mu(\zeta)}{(\varphi(\zeta) - a)^{n+1}} \qquad (n \geq 0). \tag{1.1.3}$$

Proof. Let $B = B(a,r)$ be an open ball in Ω. We then have the inequality

$$|\varphi(\zeta) - a| \geq r \qquad (\zeta \in X)$$

whence

$$\left| \frac{z-a}{\varphi(\zeta) - a} \right| \leq \frac{|z-a|}{r} < 1 \qquad (z \in B).$$

Fix $z \in B$. The above inequality shows that the series $\sum_{n=0}^{\infty} \left((z-a)/(\varphi(\zeta)-a)\right)^n$ converges uniformly in $\zeta \in X$. An easy computation shows that the series equals $(\varphi(\zeta)-a)/(\varphi(\zeta)-z)$. We thus have

$$\frac{1}{\varphi(\zeta) - z} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}} \tag{$*$}$$

with the convergence being uniform in ζ. Let δ be the distance from z to the closed set $\mathbf{C} \smallsetminus \Omega$. Since $|(\varphi(\zeta) - z)^{-1}| \leq 1/\delta$ for every $\zeta \in X$, the integrand in (1.1.1) is in $L^1(\mu)$ and hence $f(z) \in \mathbf{C}$ for all $z \in B$. The series $(*)$ converges uniformly in

$\zeta \in X$, whence the integral of the series passes through the summation sign, and we have:

$$\int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z} = \sum_{n=0}^{\infty} c_n (z-a)^n$$

with c_n satisfying (1.1.3). This proves the Theorem. $\square$

2. **Path Integrals**

A *curve* in $\mathbf{C}$ is a continuous map $\gamma \colon [a,b] \to \mathbf{C}$. The complex numbers $\gamma(a)$ and $\gamma(b)$ are called the end points of the curve γ. A *path* in $\mathbf{C}$ (in this course) will mean a curve $\gamma \colon [a,b] \to \mathbf{C}$ which is *piecewise smooth*. By this we mean that there is a partition of $[a,b]$

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that γ is differentiable on $[t_{i-1}, t_i]$, $i = 1, \ldots, n$, with the appropriate one sides derivatives existing at the end points of the interval, and the derivative function on $[t_{i-1}, t_i]$ is continuous.

The image of γ will be denoted γ^*. In other words $\gamma^* = \gamma([a,b])$.

If f is a continuous complex-valued function on a subset of $\mathbf{C}$ containing γ^*, then the path integral $\int_\gamma f(z)dz$ is defined as

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

We regard two paths γ and σ, with the same end points, equivalent if $\gamma^* = \sigma^*$ and

$$\int_\gamma f(z)dz = \int_\sigma f(z)dz$$

for all f continuous on γ^*. In particular re-parameterisations of γ give rise to equivalent paths. In greater detail, if $\varphi \colon [\alpha, \beta] \to [a,b]$ is a smooth diffeomorphism such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$ (the second condition is a consequence of the other hypotheses), then γ and $\gamma \circ \varphi$ are equivalent. A diffeomorphism is a smooth map which is bijective and whose derivatives are non-zero (with an obvious interpretation at the end points – namely the suitable one-sided derivative is non-zero).

3. **Cauchy-Riemann Equations**

Let $f \colon \Omega \to \mathbf{C}$ be a function on a non-empty open set Ω of $\mathbf{C}$, which has first partial derivatives, and whose real and Imaginary parts are u and v respectively. Then f is said to satisfy *the Cauchy Riemann equations* if

$$(*) \qquad f_x = -i f_y,$$

or in an equivalent form, if u and v satisfy

$$(**) \qquad u_x = v_y, \quad u_y = -v_x.$$

We proved the following in class:

Theorem 3.1. *Let $f \colon \Omega \to \mathbf{C}$ function on a region Ω, and $f = u_i v$ its decomposition into real and imaginary parts.*

(a) *If f is analytic, then f satisfies the Cauchy-Riemann equations.*
(b) *If f is C^1 and satisfies the Cauchy-Riemann equations, then f is analytic on Ω.*

As an easy consequence, we can prove the following

Proposition 3.1.1. *Suppose* $f\colon \Omega \to \mathbf{C}$ *is a continuous function on a region* Ω *and for any path* $\gamma\colon [a,\, b] \to \Omega$ *the integral*

$$\int_\gamma f(z)dz$$

depends only upon the end points a *and* b*. Then* f *has a primitive (i.e., an anti-derivative)* F *on* Ω*.*

Proof. Pick a point z_0 in Ω. For $z \in \Omega$ define

$$F(z) = \int_\gamma f(\zeta)d\zeta$$

where γ is any path joining z_0 to z.

For h and k non-zero real numbers, define paths γ_h and σ_k to be the horizontal and vertical paths with speed 1 from z to $z+h$ and z to $z+ik$ respectively. For example, if h is positive, γ_h is the map from $[0,\, h]$ to $\mathbf{C}$ given by $t \mapsto t+h$, and if h is negative, γ_h is the map $t \mapsto t-h$ on $[0,\, -h]$. If $k > 0$, σ_k is the map $t \mapsto z+it$ on $[0,\, k]$. In any case

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{\gamma_h} f(\zeta)d\zeta = \frac{1}{h}\int_0^h f(z+t)dt$$

and

$$\frac{F(z+ik)-F(z)}{k} = \frac{1}{k}\int_{\sigma_k} f(\zeta)d\zeta = \frac{i}{k}\int_0^k f(z+it)dt.$$

If x and y are the standard co-ordinates along real and imaginary axes, letting the (real) variables h and k go to zero in the above identities and applying the (real) fundamental theorem of Calculus we obtain

$$\frac{\partial F}{\partial x}(z) = f(z)$$

and

$$\frac{\partial F}{\partial y}(z) = if(z).$$

In other words F satisfies the Cauchy-Riemann equations. Morever, since f is continuous, F is C^1. By Theorem 3.1, F is analytic, whence $F' = F_x = f$. □