LECTURE 19

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Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

1. Continuing analytic functions across boundaries

1.1. **Continuing across line segments.** Suppose Ω is a region and L is a (non empty) line segment Ω which is a closed subset Ω. It is immaterial whether L is a closed line segment or an open line segment, or anything in between, e.g., $L = (0, 1]$ is a closed subset of $\Omega = \{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$. Let $\Omega' = \Omega \smallsetminus L$. Suppose $f \colon \Omega \to \mathbf{C}$ is a continuous function such that $f(z)$ is analytic on Ω'. If L is a singleton set, then by the Riemann removable singularity theorem, f is analytic on Ω. This remains true even when L has positive length as the following argument shows.

First, without loss of generality, we may assume L is a horizontal line segment, by applying the transformation $z \mapsto e^{i\theta}z$ for a suitable real number θ on the region Ω. Let R be any (closed, bounded) rectangle in Ω with sides parallel to the real and imaginary axes. We claim that

$$\int_{\partial R} f(z)dz = 0.$$

If L is part of one of edges of R, say the bottom edge, then for $\eta > 0$ let R_η be the rectangle whose bottom edge is at a distance η from the bottom edge of R, and whose right and left edges follow the right and left edges of R, until a distance η from the bottom edge of R, and whose top edge agrees with the top edge of R. The length of the bottom edge of R equals that of the bottom edge of R_η. Let this be denoted ℓ. Then, with the bottom left corner of R being $z_0 = a + ib$, we have

$$\begin{aligned}
\left| \int_{\partial R} f(z)dz - \int_{\partial R_\eta} f(z)dz \right| &= \left| \int_a^{a+\ell} f(x+ib)dx - \int_a^{a+\ell} f(x+i(b+\eta))dx \right| \\
&\quad + \left| \int_b^{b+\eta} f(a+\ell+iy)dy - \int_b^{b+\eta} f(a+iy)dy \right| \\
&= \int_a^{a+\ell} |f(x+ib) - f(x+i(b+\eta))|dx \\
&\quad + \int_b^{b+\eta} |f(a+\ell+iy) - f(a+iy)|dy \\
&\leq \int_a^{a+\ell} |f(x+ib) - f(x+i(b+\eta))|dx + 2\eta \sup_{z\in R} |f(z)|
\end{aligned}$$

Since f is uniformly continuous on the compact set R, given $\epsilon > 0$ we can find $\eta > 0$ such that

$$(*) \qquad |f(z) - f(w)| < \frac{\epsilon}{2\ell} \qquad (z, w \in R, |z - w| < \eta),$$

and such that

$$(**) \qquad \eta < \frac{\epsilon}{4 \sup_{z \in R} |f(z)|}.$$

For η satisfying $(*)$ and $(**)$ we see that

$$\begin{aligned} \left| \int_{\partial R} f(z)dz - \int_{\partial R_\eta} f(z)dz \right| &\le \int_a^{a+\ell} |f(x+ib) - f(x+i(b+\eta))|dx + 2\eta \sup_{z \in R} |f(z)| \\ &< \frac{\epsilon}{2\ell}\ell + 2 \sup_{z \in R} |f(z)|\eta \qquad (\text{via } (*)) \\ &< \epsilon \qquad (\text{via } (**)). \end{aligned}$$

Thus

$$\int_{\partial R} f(z)dz = \lim_{\eta \to 0} \int_{\partial R_\eta} f(z)dz.$$

But by Cauchy-Goursat $\int_{\partial R_\eta} f(z)dz = 0$ for $\eta > 0$. Hence

$$\int_{\partial R} f(z)dz = 0$$

in this case. The same argument works when L is part of the top horizontal line segment of R.

Next if L intesects R, say at $\alpha + i\beta$, then we let σ be the horizontal line segment from $a + i\beta$ to $a + \ell + i\beta$, and let R_1 and R_2 be the rectangles obtained from R by dividing it along the horizontal line segment σ. In this case

$$\int_{\partial R} f(z)dz = \int_{\partial R_1} f(z)dz + \int_{\partial R_2} f(z)dz = 0.$$

Finally, if L does not intersect R, then by the Cauchy-Goursat theorem we have $\int_{\partial R} f(z)dz = 0$.

Thus in all cases we have $\int_{\partial R} f(z)dz = 0$. Now if D is any open disc in Ω, say with centre $a \in \Omega$, then we can find a primitive F for f in D by defining $F(z)$ to be the integral of f along the path which starts at a moves parallel to the real axis first and then moves parallel to the imaginary axis to end at z. From the fact that the integral of f along the boundary of any rectangle in Ω (with edges parallel to the axes) is zero, it is clear that $F(z)$ can also worked out by integrating f along a path which starts at a, and moves first parallel to the imaginary axis and then along an appropriate horizontal segment. Standard arguments, exactly as in the proof of Cauchy-Goursat in a disc, show that F satisfies the Cauchy-Riemann equations and is C^1 (in fact $F_x = f$, $F_y = if$ as is easily verified). Hence F is analytic and so its derivative is f. This means f is analytic on Ω.

1.2. **Continuing past circular arcs.** Suppose the closed set L in the previous section is not a line segment but an arc of a circle C, and suppose f is continuous on Ω and analytic on $\Omega' = \Omega \smallsetminus L$. Then too f is analytic on Ω. That is worked as follows. Let $a \in C$. Consider $\Omega^* = \Omega \smallsetminus \{a\}$ and $L^* = L \smallsetminus \{\}$. The univalent function $z \mapsto (z-a)^{-1}$ on Ω^* transforms L^* into a line segment. Hence applying the previous argument, the function $f(z)$ is holomorphic on Ω^*. If $a \notin \Omega$ we are done. If not, then the Riemann removable singularity theorem applies and again f is holomorphic on Ω.

The conclusions of the previous two subsections can be summarised as follows:

Proposition 1.2.1. *Suppose Ω is a region and L a closed subset of Ω which is either a line segment or a circular arc. If $f\colon \Omega \to \mathbf{C}$ is a continuous function which is holomorphic on $\Omega \smallsetminus L$ then f is holomorphic on Ω.*

2. Harmonic functions and the averaging property

We know that if u is a harmonic function on a region Ω then u has the averaging property, i.e.,

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta \tag{2.1}$$

for all $a \in \Omega$ and all $r > 0$ such that the closed disc $\overline{B}(a,r)$ is contained in Ω.

The converse is also true for continuous real-valued functions on Ω with a *local averaging property*, a notion which we define now.

Definition 2.1.1. A real-valued function u on Ω is said to have a *local averaging property* if for each $a \in \Omega$ we have a positive real number $\rho(a)$ such that (2.1) is true for all $0 < r \leq \rho(a)$.

We then have the following result

Theorem 2.2. *Let Ω be a region. A function $u\colon \Omega \to \mathbf{R}$ is harmonic if and only if it is continuous and has the local averaging property.*

It is clear that if u is harmonic it is continuous and has the local averaging property. We will prove the converse in a later lecture.

3. Symmetric Regions and Schwarz's Reflection Principle

3.1. **Notations.** Let $\mathfrak{h}$ denote the open upper half-plane and $\mathfrak{h}^-$ the open lower half-plane. For a region Ω set $\Omega^+ = \Omega \cap \mathfrak{h}$, $\Omega^- = \Omega \cap \mathfrak{h}^-$, and $\widehat{\Omega} = \{z \in \mathbf{C} \mid \bar{z} \in \Omega\}$. If $f\colon \Omega \to \mathbf{C}$ is a map, let

$$\widehat{f}\colon \widehat{\Omega} \to \mathbf{C}$$

be the map given by

$$z \mapsto \overline{f(\bar{z})}.$$

Recall from Problem 1 of HW 1 that f is holomorphic on Ω if and only if $\widehat{f}$ is holomorphic on $\widehat{\Omega}$. Note that $\widehat{\widehat{\Omega}} = \Omega$ and $\widehat{\widehat{f}} = f$.

Definition 3.1.1. A region Ω is said to be *symmetric* if $\widehat{\Omega} = \Omega$.

Lemma 3.1.2. *Suppose Ω is a symmetric region, L a line segment of positive length in $\Omega \cap \mathbf{R}$ and $f(z)$ a holomorphic function on Ω which takes real values on L. Then $f = \widehat{f}$.*

Proof. The functions f and $\widehat{f}$ are both holomorphic on Ω and agree on L. By the identity principle we are done. $\square$

3.2. **Schwarz's Reflection Principle.** Here is a weak version of the principle.

Theorem 3.2.1 (Weak Schwarz's Reflection Principle)**.** *Suppose Ω is a symmetric region, Ω_1 the set of points in Ω with non-negative imaginary parts, and $L = \Omega \cap \mathbf{R}$. If $f\colon \Omega_1 \to \mathbf{C}$ is a continuous function that is holomorphic on Ω^+ and takes real values on L, then f can be extended to a unique holomorphic function g on Ω. The extended function g satisfies the relation $g = \widehat{g}$.*

Proof. Uniqueness of g follows from the identity principle. The identity $g = \widehat{g}$ is a direct consequence of Lemma 3.1.2. Existence is the only thing that remains to be proved. Let $g\colon \Omega \to \mathbf{C}$ be defined by:

$$g(z) = \begin{cases} f(z), & z \in \Omega_1, \\ \widehat{f}(z), & z \in \Omega^-. \end{cases}$$

Note that $\Omega_1 = \Omega^+ \cup (\Omega \cap \mathbf{R})$ and hence the above does define a function on Ω. A little thought shows that Ω is continuous on Ω. Moreover it is holomorphic on $\Omega \smallsetminus L = \Omega^+ \cup \Omega^-$. Hence by Proposition 1.2.1, the map g is holomorphic on all of Ω. $\square$

The above version of Schwarz's Reflection Principle is enough for most purposes and is often called the Schwarz Reflection Principle. There is however a stronger form whose proof is independent of the above proof.

Theorem 3.2.2 (Schwarz's Reflection Principle)**.** *Let Ω and Ω_1, L be as in the hypothesis of Theorem* 3.2.1*. Let $f\colon \Omega^+ \to \mathbf{C}$ be a holomorphic function such that if $x \in L$ and $\{z_n\}$ is a sequence of points in Ω^+ converging to x, then* $\lim_{n\to\infty} \operatorname{Im}(z_n) = 0$*. Then f can be extended to a unique holomorphic function g on Ω. The extended function g satisfies the relation $g = \widehat{g}$.*

Proof. Let $f = u + iv$ be the decomposition of f into its real and imaginary parts. Define $v^*\colon \Omega \to \mathbf{C}$ by

$$v^*(z) = \begin{cases} v(z), & z \in \Omega^+, \\ 0, & z \in \Omega \cap \mathbf{R} = \Omega_1 \cap \mathbf{R}, \\ -v(\bar{z}), & z \in \Omega^-. \end{cases}$$

It is easy to see that v^* is continuous. We claim that v^* has the local averaging property on Ω. Since $v^*|_{\Omega^+} = v$, v^* is harmonic on Ω^+, and hence has the averaging property on Ω^+. Further $v^*(z) = \operatorname{Im}(\widehat{f}(z))$ for $z \in \Omega^-$, and hence v^* is harmonic on Ω^-. In particular it has the averaging property there. Now suppose $a \in L$. We can find a positive real number $\rho(a)$ such that the closed disc $|z - a| \leq \rho(a)$ lies in Ω. Let $D_a = B(a, \rho(a))$. It is easy to see that (2.1) is satisfied by v^* for $0 < r < \rho(a)$. Indeed a change of variables shows that $\int_\pi^{2\pi} v(a + re^{-i\theta})d\theta = \int_0^\pi v(a + re^{i\theta})d\theta$, whence $(2\pi)^{-1}\int_0^{2\pi} v^*(a + re^{i\theta})d\theta = 0 = v^*(a)$ for $0 < r \leq \rho(a)$. This proves the claim. By Theorem 2.2 we see that v^* is harmonic.

Let $-u_a^*$ be a harmonic conjugate of v^* on D_a, where $a \in L$ and D_a is as above. Then $g_a = u_a^* + iv^*$ is holomorphic on D_a. Moreover on D_a^+, $g_a - f$ takes real values and hence must be a real constant for it is not an open map. We can pick our harmonic conjugate $-u_a^*$ in such a way that $c_a = 0$. Then $g_a|_{D_a^+} = f|_{D_a}$, and let us do so. Since D_a is a symmetric region and g takes real values on $D_a \cap \mathbf{R}$, by Lemma 3.1.2 we have $\widehat{g_a} = g_a$. It follows that $g_a(z) = \widehat{f}(z)$ for $z \in D_a^-$.

By the identity principle, for $a, b \in L$, $g_a|_{D_a \cap D_b} = g_b|D_a \cap D_b$. If $U = \cup_{a \in L} D_a$, then from what we've seen, the g_a patch to give a holomorphic function $g_U : U \to \mathbf{C}$ such that g_U agrees with f on $\Omega^+ \cap U$ and with $\widehat{f}$ on $\Omega^- \cap U$. Since f, g_U, and $\widehat{f}$ agree on the intersections of their domains, one checks easily that they glue to give a holomorphic function g on Ω. As before uniqueness follows from the identity principle. The property $\widehat{g} = g$ is true because g is real-valued on L (see Lemma 3.1.2). □