LECTURE 18

Date of Lecture: March 13, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

1. **The Riemann Mapping Theorem**

Let Δ denote the unit disc around $0 \in \mathbf{C}$. Last lecture (Lecture 17) we showed the following (see Lemma 2.2.1 of *loc.cit.*):

Let U be a simply connected open subset of Δ such that $0 \in U$ and $U \neq \Delta$. Then there exists a univalent function $f\colon U \to \Delta$ such that $f(0) = 0$ and $|f'(0)| > 1$.

A little thought shows that if f in the above has maximal derivative (in absolute value) at 0 amongst all univalent functions on U taking values in Δ which take 0 to 0, then $f(U) = \Delta$. In other words if f in the result in italics above is such $|f'(0) = \sup_g |g'(0)|$, where the supremum is taken over univalent maps $g\colon U \to \Delta$ with $g(0) = 0$, then $f(U) = \Delta$. If not, then we have a univalent map $h\colon f(U) \to \Delta$ with $h(0) = 0$ and $|h'(0)| > 1$. Then $g = h \circ f$ is such that g is univalent on U, $g(0) = 0$ and $|g'(0)| > |f'(0)|$, a contradiction. This leads us to the question, is there a f as in Lemma 2.2.1 of Lecture 17 with maximal possible derivative (in absolute value) at 0? This an example of an extremal problem. We examine a slightly more general problem in the subsection that follows.

1.1. **An Exremal Problem.**

We fix a *simply connected* region Ω such that $\Omega \neq \mathbf{C}$, as well a point $z_0 \in \Omega$. Set

$$\mathscr{F} = \{f\colon \Omega \to \Delta \mid f \text{ is univalent}\}.$$

The extremal problem we pose is: *Show there exists $f \in \mathscr{F}$ such that $f'(z_0) > 0$ and if $g \in \mathscr{F}$ then $|g'(z_0)| \leq f'(z_0)$.*

Any solution of this extremal problem is called a Riemann mapping for Ω at z_0. As we will see a solution exists and it is unique. First let us observe the following.

Lemma 1.1.1. *Suppose V is a simply connected region of $\mathbf{C}$ such that $0 \notin V$. Let $g_i\colon V \to \mathbf{C}$, $i = 1, 2$ be the two branches of the square root function V (which exist because of our hypotheses on V). Then $g_1(V) \cap g_2(V) = \emptyset$.*

Proof. Suppose $c \in g_1(V) \cap g_2(V)$. Then $c = g_1(a)$ and $c = g_2(b)$ for some a and b in V. Squaring, we see that $a = c^2 = b$. Thus $g_1(a) = g_2(a)$, i.e., $g_1(a) = -g_1(a)$. This means $g_1(a) = 0$, whence upon squaring $a = 0$, contradicting the fact that $0 \notin V$. □

1.1.2. *Solution to the Extremal Problem.* Let us first show that $\mathscr{F} \neq \emptyset$. Let $a \in \mathbf{C} \smallsetminus \Omega$. Then $h(z) = z - a$ is nowhere vanishing and univalent on Ω. Let $V = h(\Omega)$. Then V is simply connected and $0 \notin V$. Hence if $\varphi\colon V \to \mathbf{C}$ is a branch of $\sqrt{z}$ on V, then by Lemma 1.1.1, $\varphi(V) \cap (-\varphi)(V) = set$. Let D be disc of radius ρ in $(-\varphi)(V)$ centred at $w_0 \in (-\varphi)(V)$. Then, clearly $z - w_0 > \rho$ for all $z \in V = h(\Omega)$. It follows that $\psi(z) = \rho/(z - w_0)$ is univalent map on $\varphi(V)$ which takes values in Δ.

If $H(z) = \psi \circ \varphi \circ h$, then H is univalent on Ω and takes values in Δ. Thus $H \in \mathscr{F}$, whence $\mathscr{F} \neq \emptyset$.

Let $M = \sup_{f \in \mathscr{F}} |f'(z_0)|$. Note that if $\varepsilon > 0$ is so small that the disc $|z - z_0| \leq \varepsilon$ lies in Ω, then we have

$$|f'(z_0)| = \frac{1}{2\pi}\left|\int_{|z-z_0|=\varepsilon} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta\right| \leq \frac{1}{2\pi}\frac{1}{\varepsilon^2}(2\pi\varepsilon) \leq \frac{1}{\varepsilon}, \qquad (f \in \mathscr{F}).$$

Thus $M < \infty$. We can find a sequence $\{f_n\}$ in $\mathscr{F}$ such that $\{|f'(z_0)|\}$ converges to M. Since $\mathscr{F}$ is locally bounded, in fact it is globally bounded by 1 since its members take values in Δ, by Montel's Theorem $\{f_n\}$ has a convergent subsequence $\{f_{n_k}\}$ which converges uniformly on compact sets. Let $f = \lim k \to \infty f_{n_k}$. Then we know that $f \in \mathscr{H}(\Omega)$. Since each f_{n_k} is univalent, by Hurwitz's Theorem, either f is a constant or it is univalent. Further, we know that $f'_{n_k} \to f'$ uniformly on compact sets as $k \to \infty$. This means $|f'(z_0)| = M > 0$. This means f cannot be a constant. Hence f is univalent. It clearly takes values in Δ.

We claim $f(z_0) = 0$. Suppose $f(z_0) = b$. Note $b \in \Delta$. Consider the function $\Phi_b(z) = \dfrac{z - b}{1 - \bar{b}z}$. Then $\Phi_b \circ f \in \mathscr{F}$, and $(\Phi_b \circ f)(z_0) = 0$. Moroever, $\Phi_b'(b) = \dfrac{1}{1 - |b|^2}$. Hence we have

$$\frac{M}{1 - |b|^2} = |\Phi_b'(b)||M| = |\Phi_b'(b)||f'(z_0)| = |(\Phi_b \circ f)(z_0)| \leq M,$$

giving,

$$\frac{1}{1 - |b|^2} \leq 1.$$

The above inequality implies $b = 0$ since we know $|b| < 1$. Thus $f(z_0) = 0$ and $|f'(z_0)| = M$. It follows that $f'(z_0) = e^{i\theta}M$ for some real θ. Replacing f by $e^{-i\theta}f$ if necessary, we see that f solves the Extremal Problem.

Lemma 1.1.2. *Let f be the solution to the above Extremal problem. Then $f(\Omega) = \Delta$.*

Proof. If not, then $U = f(\Omega)$ is a simply connected region in Δ which is not equal to Δ, whence we have a univalent map $g\colon U \to \Delta$ such that $|g'(0)| > 1$. The composite $g \circ f$ is in $\mathscr{F}$ and $|(g \circ f)'(0)| > |f'(0)|$, a contradicton. □

1.2. **The Riemann Mapping Theorem.** Here is the statement

Theorem 1.2.1 (The Riemann Mapping Theorem)**.** *Let Ω be a simply connected region such that $\Omega \neq \mathbf{C}$, and let z_0 be a point in Ω. Then there exists a unique univalent* onto *map*

$$f\colon \Omega \to \Delta$$

such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Remark: Note that f as in the Theorem gives us a biholomorphism between Ω and Δ. In particular, Ω and Δ are homeomorphic, whence Ω is classically simply connected.

Proof. We have already seen the existence of such an f (see Lemma 1.1.2 to see that the solution to the extremal problem in the last subsection maps Ω surjective on to Δ). It only remains to prove uniqueness. So suppose g is another biholomorphism from Ω to Δ which vanishes at z_0 and such that $g'(z_0) > 0$. Then by Schwarz's Lemma there exists $\theta \in \mathbf{R}$ such that $(f \circ g^{-1})(z) = e^{i\theta} z$. Now $e^{i\theta} = (f \circ g^{-1})'(0) = f'(0)/g'(0) > 0$. Hence $e^{\theta} = 1$, whence $f = g$. $\square$

Corollary 1.2.2. *A region Ω is classically simply connected if and only if it is simply connected.*

Proof. We have seen earlier in the course that classically simply connected regions are simply connected. The Riemann Mapping Theorem gives the converse (the case $\Omega = \mathbf{C}$ is trivially classically simply connected). $\square$