LECTURE 17

Date of Lecture: March 8, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

1. Hurwitz's Theorem and consequences

1.1. **Uniform convergence of derivatives.** Let Ω be a region and suppose $f_n \to f$ as $n \to \infty$ uniformly on compact subsets of Ω with each f_n holomorphic on Ω. We have seen, via an application of Morera's theorem, that f is holomorphic. What about the convergence of the derivatives f_n'? It turns out that $f_n' \to f'$ uniformly on compact sets.

Lemma 1.1.1. *Let $\{f_n\}$ be a sequence in $\mathscr{H}(\Omega)$ converging uniformly on compact subsets of Ω to $f \in \mathscr{H}(\Omega)$. Then $\{f_n'\}$ converges uniformly on compact subsets of Ω to f'.*

Proof. It is enough to prove that $f_n' \to f'$ (as $n \to \infty$) on compact subsets of Ω of the form $K = \overline{B}(a,r)$. We can find $R > r$ such that $\overline{B}(a,R) \subset \Omega$, because the distance from K to $\mathbf{C} \subset \Omega$ is greater than 0. Thus $K \subset \overline{B}(a,R)$. Let C be the bounding circle of $\overline{B}(a,R)$. We have

$$(*) \qquad f_n'(z) - f'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)-f(\zeta)}{(\zeta - z)^2}d\zeta \qquad (z \in K).$$

Given $\eta > 0$ we have $N_\eta \geq 1$ such that $|f_n(\zeta) - f(\zeta)| < \eta$ for all $z \in K$ and all $n \geq N_\eta$. From $(*)$ we get for $z \in K$ and $n \geq N_\eta$

$$\begin{aligned} |f_n'(z) - f'(z)| &\leq \frac{1}{2\pi}\int_C \frac{|f_n(\zeta)-f(\zeta)|}{|\zeta - z|^2}|d\zeta| \\ &< \frac{1}{2\pi}\frac{\eta}{(R-r)^2}(2\pi R) \\ &= \frac{R\eta}{(R-r)^2}. \end{aligned}$$

Thus given $\epsilon > 0$ and picking $\eta = (R-r)^2\epsilon/R$ we see that there exists $N \geq 1$ (choose $N = N_\eta$) such that

$$|f_n'(z) - f'(z)| < \epsilon, \qquad (z \in K,\, n \geq N)$$

giving the required result. ☐

1.2. **Hurwitz's Theorem.** Recall that a one-to-one holomorphic function is called *univalent*. The German word *schlicht* is also used. Our interest is as much in Hurwitz's theorem regarding the convergence of nowhere vanishing holomorphic functions, as on its important corollary regarding the convergence of univalent functions.

Theorem 1.2.1 (Hurwitz's Theorem)**.** *Let Ω be a region and $\{f_n\}$ a sequence of nowhere vanishing holomorphic functions on Ω converging uniformly on compact subsets of Ω to a function f. Then either f is identically zero on Ω or it is nowhere vanishing on Ω.*

Proof. Suppose f is not identically zero. Then its zeros are isolated. Let $a \in \Omega$. We can find a disc $D_a = B(a, r)$ in Ω such that $\overline{D_a} \subset \Omega$ and the bounding circle C of $\overline{D_a}$ does not contain any zeros of f. By Cauchy's Theorem (or by the Argument Principle) we have

$$(\dagger) \qquad \int_C \frac{f_n'(z)}{f_n(z)} dz = 0 \qquad (n \in \mathbb{N}).$$

Let

$$m := \inf_{\zeta \in C} |f(\zeta)|.$$

Since C is compact and f does not vanish on C there exists, we see that $m > 0$. Next let

$$M := \max\{\sup_{\zeta \in C} |f|, \sup_{\zeta \in C} |f'|\}.$$

Since $\{f_n\}$ converges to f uniformly on C and $\{f_n'\}$ converges to f' uniformly on C we have an integer $N \geq 1$ such that $|f_n(z)| > m/2$, $|f_n(z)| < 2M$, and $|f_n'(z)| < 2M$ for $n \geq N$ and $z \in C$. We therefore have

$$\left| \frac{f_n'(z)}{f_n(z)} - \frac{f'(z)}{f(z)} \right| \leq \left| \frac{f(z)f_n'(z) - f'(z)f_n(z)}{f_n(z)f(z)} \right| \leq 8\frac{M^2}{m^2} \qquad (z \in C).$$

The Dominated Covergence Theorem can now be used to give, via ($\dagger$),

$$\int_C \frac{f'(z)}{f(z)} dz = \lim_{n\to\infty} \int_C \frac{f_n'(z)}{f_n(z)} dz = 0.$$

By the Argument Principle, it follows that $f(z)$ is nowhere vanshing on D_a. Since D_a's cover Ω, we are done. $\square$

Corollary 1.2.2. *Suppose $\{f_n\}$ is a sequence of univalent functions on a region Ω which converges uniformly on compact subsets of Ω to a function f. Then either f is a constant function or f is univalent.*

Proof. Suppose f is not constant. Then f' is not identically zero. Since $\{f_n'\}$ is sequence of nowhere vanishing holomorphic functions converging uniformly on compact subsets of Ω to f', by Hurwitz's theorem this means f' is nowhere vanishing. Let $z_0 \in \Omega$ and let $\Omega' = \Omega \setminus \{z_0\}$. Set $h_n(z) = f_n(z) - f(z_0)$ and $h(z) = f(z) - f(z_0)$. Then h_n is nowhere vanishing on Ω' and converges uniformly on compact subsets of Ω' to h. So either h is identically zero or it is nowhere vanishing via another application of Hurwitz's theorem. Now $h' = f'|_{\Omega'}$ and f' is nowhere vanishing. Since $h' \neq 0$, the function h cannot be identically zero as we just argued. This means $f(z) \neq f(z_0)$ if $z \neq z_0$, whence f is univalent. $\square$

2. **Revisiting Schwarz's Lemma**

The main point of this section is to understand the failure of Schwarz's Lemma when the hypothesis is weakened to allow holomorphic functions from an open subset of $B(0, 1)$ to $B(0, 1)$.

2.1. **Simple connectedness and univalent maps.** To what extent is simple connectedness preserved under a univalent map? Recall that if Ω is a region and $f\colon \Omega \to \mathbf{C}$ is univalent then as f is non-constant (on any open subset of Ω) it is an open map. In particular it defines a homeomorphism from Ω to $f(\Omega)$. Moreover, the inverse map $g\colon f(\Omega) \to \Omega$ is necessarily holomorphic. One way to see this is via problem 6 (b) and (c) of HW 6. Equivalently, the inverse-function theorem for smooth maps tells us that the (several real variables) derivative of g, is the 2×2 matrix $\frac{1}{|f'|^2}\left(\begin{smallmatrix} v_y & -u_y \\ -v_x & u_x \end{smallmatrix}\right)$. By Cauchy-Riemann this is of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, whence by Lemma 2.1.3 of Lecture 14-15 we see that g is conformal, whence it is holomorphic (see Proposition 2.2.3 of *loc.cit.*). Thus the univalent map f gives us a *biholomorphism* $f\colon \Omega \to f(\Omega)$. It therefore seems reasonable to expect $f(\Omega)$ to be simply connected if Ω is, and the following result shows that this is indeed so.

Lemma 2.1.1. *Let $\varphi\colon \Omega \to \mathbf{C}$ be a univalent map on a simply connected region Ω. Then the image $\varphi(\Omega)$ is a simply connected region.*

Proof. It is clear that $V := \varphi(\Omega)$ is a region, since it a non-empty open set which must be connected. Let $\psi\colon V \to \Omega$ be the inverse of φ. As we just saw (above the statement of the Lemma), ψ is holomorphic. According to Theorem 3.2.1 of Lecture 12-13, a region U is simply connected if and only if every holomorphic function on it has a primitive. To that end, let $g\colon V \to \mathbf{C}$ be a holomorphic function on $V = f(\Omega)$. We have to show g has a primitive. Let $f = g \circ \varphi$. On Ω we have the holomorphic function $p(z) = f(z)\varphi'(z)$. Since Ω is simply connected the function $p(z)$ has a primitive, say $F(z)$, on Ω. Define $G = F \circ \psi$. Then

$$\begin{aligned} G'(w) = F'(\psi(w))\psi'(w) &= f(\psi(w))\varphi'(\psi(w))\psi'(w) \\ &= g(w)(\varphi \circ \psi)'(w) \\ &= g(w). \end{aligned}$$

Thus g has a primitive, namely G. □

2.2. **Univalent functions on open subsets of Δ.** Throughout, Δ will denote $B(0,1)$, the unit disc centred at 0. The bounding circle $\{|z| = 1\}$ will be denoted C.

Recall that if $b \in \Delta$ then the map

$$\Phi_b(z) := \frac{z - b}{1 - \bar{b}z}$$

is bi-holomorphic on the compact set $\overline{\Delta}$ and $\Phi_b(C) = C$. The inverse of Φ_b on $\overline{\Delta}$ is Φ_{-b} as is easily verified.

Here is the main result.

Lemma 2.2.1. *Let U be a simply connected open subset of Δ such that $0 \in U$ and $U \neq \Delta$. Then there exists a univalent function $f\colon U \to \Delta$ such that $f(0) = 0$ and $|f'(0)| > 1$.*

Remark: Compare this to the Schwarz's Lemma.

Proof. Since $U \neq \Delta$ we can find $b \in \Delta$, such that $b \notin U$. Then Φ_b is nowhere vanishing on U, and hence $U_1 := \Phi_b(U)$ is simply connected by Lemma 2.1.1. Note that $-b = \Phi_b(0)$ lies in U_1. Since U_1 is simply connected and does not contain 0, one can has a branch of $\sqrt{z}$ on U_1 (see Problem 4 (c) of HW 6). Let $g\colon U_1 \to \Delta$

be such a branch, and let us agree to write $\sqrt{z}$ for $g(z)$ when $z \in U_1$. Note that $g \circ \Phi_b(0) = \sqrt{-b}$. Moreover g is univalent on U_1 for if $g(d) = g(e)$ then $\sqrt{d} = \sqrt{e}$, whence upon squaring, we get $d = e$. Let $U_2 = g(U_1)$. Let $V = \Phi_{-\sqrt{-b}}(U_2)$ and set

$$f = \Phi_{\sqrt{-b}} \circ g \circ \Phi_b.$$

Then $f\colon U \to \Delta$ is univalent, $f(0) = 0$, $f(U) = V$.

Next suppose

$$\Psi\colon \overline{\Delta} \to \overline{\Delta}$$

is the map given by

$$\Psi := \Phi_{-b} \circ h \circ \Phi_{-\sqrt{-b}}$$

where $h\colon \Delta \to \Delta$ is the map $z \mapsto z^2$. Then it is easy to see that

- $\Psi(V) = U$,
- $f \circ (\Psi|_V)(w) = w$ for all $w \in V$, and
- $\Psi \circ f(z) = z$ for all $z \in U$.

Note that $\Psi(0) = 0$ so that Ψ satisfies the hypotheses of Scharz's Lemma. However, since h is not one-to-one on Δ, Ψ is not one-to-one on Δ. This means

$$(**) \qquad |\Psi'(0)| < 1$$

(otherwise $|\Psi'(0)| = 1$ and by Schwarz's Lemma, Ψ would be one-to-one). The chain rule for differentiation gives us

$$1 = \left| \frac{dz}{dz}|_{z=0} \right| = |(\Psi \circ f)'(0)| = |\Psi'(f(0))||f'(0)| = |\Psi'(0)||f'(0)|$$

whence by $(**)$ we get

$$|f'(0) = |\Psi'(0)|^{-1} > 1.$$

□