LECTURE 16

Date of Lecture: March 6, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

1. Topology on the space of holomorphic functions

Last time we showed that if a non-empty family of holomorphic functions on a region is locally bounded then it is uniformly equicontinuous on compact subsets of that region. The two notions are actually equivalent, and equivalent to a few other related notions. It is most convenient to work with a topology (coming from a metric) on the space of holomorphic functions on a region.

1.1. The space $\mathscr{H}(\Omega)$.

Suppose Ω is a region. Let $\mathscr{H}(\Omega)$ denote the space of holomorphic functions on Ω. For each compact subset K of Ω and each $f \in \mathscr{H}(\Omega)$ let us write $\|f\|_K$ for the supremum of $|f|$ on K. (In other words $\|f\|_K = \|(f|_K)\|_\infty$.) For each $n \in \mathbb{N}$, define a compact set K_n by

$$K_n := \{z \in \Omega \mid |z| \le n,\ \mathrm{dist}(z, \mathbf{C} \smallsetminus \Omega) \ge 1/n\}.$$

Then one checks easily that

- $K_1 \subset K_2 \subset \ldots K_n \subset K_{n+1} \subset \ldots$, and
- $\bigcup_{n=1}^{\infty} K_n = \Omega$,
- if K is a compact subset of Ω then $K \subset K_n$ for some $n \ge 1$.

A sequence of compact sets $\{K_n\}$ satisfying the above three properties is called an *exhaustion of Ω by compact subsets*. Let $\|f\|_n = \|f\|_{K_n}$ for $n \in \mathbb{N}$ and $f \in \mathscr{H}(\Omega)$. Define a metric d on $\mathscr{H}(\Omega)$ as follows:

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_n}{1+\|f-g\|_n} \qquad (f, g \in \mathscr{H}(\Omega)). \tag{1.1.1}$$

We have the following

Lemma 1.1.2. *The metric space $(\mathscr{H}(\Omega), d)$ is complete. A sequence $\{f_n\}$ in $\mathscr{H}(\Omega)$ converges with respect to the metric d if and only if it converges uniformly on compact subsets of Ω.*

Proof. We will first prove that convergence in $(\mathscr{H}(\Omega), d)$ implies uniform convergence on compact sets.

Let $\{f_k\}$ be a convergent sequence in $(\mathscr{H}(\Omega), d)$. Say it converges to $f \in \mathscr{H}(\Omega)$. We will show $f_k \to f$ as $k \to infty$ uniformly on each K_n. Since every compact set is contained in some K_n, this will show that $\{f_k\}$ converges uniformly on compact sets to$f$. Fix $n \in \mathbb{N}$. Given $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that

$$d(f_k, f) < \frac{1}{2^n}\frac{\epsilon}{1+\epsilon} \qquad (l, k \ge N).$$

From (1.1.1) it follows that

$$\frac{1}{2^n}\frac{\|f_k - f\|_n}{1+\|f_k - f\|_n} \le \frac{1}{2^n}\frac{\epsilon}{1+\epsilon} \qquad (k \ge N).$$

It follows that

$$\|f_k - f\|_n < \epsilon \qquad (k \ge N).$$

Thus $\{f_k\}$ converges uniformly to f on K_n.

The above proof, with obvious modifications, also shows that if $\{f_k\}$ is Cauchy in $(\mathscr{H}(\Omega), d)$, then it is Cauchy in $C(K)$ for each compact subset K of Ω where $C(K)$ is the space of continuous functions on K with supremum norm $\|\cdot\|_K$.

Now suppose $\{f_k\}$ converges uniformly on compact subsets on Ω. Let f be the limiting function. Recall f must belong to $\mathscr{H}(\Omega)$ in this case. Let $\epsilon > 0$ be given. Let $n_0 \in \mathbb{N}$ be chosen such that $\sum n = n_0 + 1^{\infty} 2^{-n} < \epsilon/2$. Since $f_k \to f$ uniformly onK_{n_0} therefore there exists $N \ge 1$ such that $\|f_k - f\|_{n_0} < 2\epsilon/(1-2\epsilon)$ for all $k \ge N$. In fact, since $K_n \subset K_{n_0}$ for all $n \le n_0$, we have $\|f_k - f\|_n < 2\epsilon/(1-2\epsilon)$ for all $k \ge N$. Thus if $k \ge N$ we have

$$\begin{aligned} d(f_k, f) &< \sum_{n=1}^{n_0} \frac{1}{2^n}\frac{\|f_k - f\|_n}{1+\|f_k - f\|_n} + \frac{\epsilon}{2} \\ &< \sum_{n=1}^{n_0} \frac{1}{2^n}\frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

The same proof with obvious modifications also shows that if $\{f_n\}$ is Cauchy in each $C(K)$, with K compact in Ω, then it is Cauchy in $(\mathscr{H}(\Omega), d)$. This proves the Lemma. □

The following is known as Montel's Theorem[1].

Theorem 1.1.3 (Montel's Theorem)**.** *Let Ω be a region and $\mathscr{F}$ a non-empty subset of $\mathscr{H}(\Omega)$. The following are equivalent:*

(a) *$\mathscr{F}$ is locally bounded.*
(b) *$\mathscr{F}$ is locally bounded and uniformly equicontinuous on compact subsets of Ω.*
(c) *Every sequence in $\mathscr{F}$ has a subsequence which converges uniformly on compact sets.*
(d) *Every sequence in $\mathscr{F}$ has a subsequence which converges in $(\mathscr{H}(\Omega), d)$.*
(e) *The closure $\overline{\mathscr{F}}$ of $\mathscr{F}$ in $(\mathscr{H}(\Omega), d)$ is compact.*

Proof. We have already proven (a) ⇒ (b). Lemma 1.1.2 yields (c) ⇔ (d) ⇔ (e).

We will now prove (b) ⇒ (c). For K a compact subset of Ω let us write $\mathscr{F}|_K$ for the collection $\{f|_K\}_{f\in\mathscr{F}}$. Let K be a compact subset of Ω. Suppose (b) is true. According to Arzela-Ascoli this is equivalent to saying $\mathscr{F}|_K$ is relatively compact in $(C(K), \|\cdot\|_K)$, i.e., the closure of $\mathscr{F}|_K$ in $C(K)$ is compact. It follows that every sequence in $\mathscr{F}$ has a subsequence (depending upon the compact set K) which converges uniformly on K. Let $\{f_n\}$ be a sequence in $\mathscr{F}$. Let $\{K_l\}_{l=1}^{\infty}$ be the exhaustion of Ω by compact subsets that we introduced earlier (to define the metric d). Let $\{g_{n,1}\}$ be a subsequence of $\{f_n\}$ such that $\{g_{n,1}\}$ converges uniformly on K_1. For $j \ge 1$, we recursively define $\{g_{n,j+1}\}$ to be a subsequence of $\{g_{n,j}\}$ such

[1]More precisely (a)⇔(c) is what is traditionally known as Montel's Theorem.

that $\{g_{n,j+1}\}$ converges uniformly on K_{j+1}. Set $f_{n_k} = g_{k,k}$. One checks easily that $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$ which converges uniformly on each K_j, whence on each compact subset of Ω. Thus (c) is true.

Finally we show that (c) $\Rightarrow$ (a). Suppose $\mathscr{F}$ satisfies (c) and let K be a compact subset of Ω. The condition (c) is equivalent to saying that the closure of $\mathscr{F}|_K$ in $(C(K), \|\cdot\|_K)$ is compact. This means $\mathscr{F}|_K$ is bounded (in fact totally bounded) in $(C(K), \|\cdot\|_K)$, and hence there exists $M_K < \infty$ such that $\|f\|_K < M_K$ for every $f \in \mathscr{F}$. In other words $\mathscr{F}$ satisfies (a). $\square$

Definition 1.1.4 (Normal Families)**.** Let Ω be a region. A non-empty family $\mathscr{F}$ of holomorphic functions on Ω is said to be a *normal family*[2] if it satisfies any of the equivalent conditions of Theorem 1.1.3.

[2]Traditionally, the definition is reserved for $\mathscr{F}$ satisfying (c) of Montel's Theorem.