LECTURES 14-15

Dates of Lectures: March 1 and 2, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

1. Rouché's Theorem

Suppose Ω is a region and Γ a cycle in Ω homologous to zero in Ω, and suppose that for $z \notin \Gamma^*$, the winding number of Γ about z is either 0 or 1. Then we say Γ *encloses* a point $a \in \Omega \smallsetminus \Gamma^*$ if $\eta(\Gamma, z) = 1$.

Theorem 1.1. *Let Ω be a region and Γ a cycle in Ω such that $\Gamma \sim 0$ (mod Ω) and such that $\eta(\Gamma, z)$ is either 0 or 1 for z not in Γ^*. Suppose $f(z)$ and $g(z)$ are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on Γ^*. Then $f(z)$ and $g(z)$ have the same number of zeros (counted with multiplicity) enclosed by Γ.*

Proof. One checks easily that if $z \in \Gamma^*$, then neither $f(z)$, nor $g(z)$, is zero. Clearly

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1$$

on Γ^*. Let $F(z) = g(z)/f(z)$. From the inequality above we see that $F(\Gamma^*) \subset B(1, 1)$. It follows that 0 lies in the unbounded component of $\mathbf{C} \smallsetminus f(\Gamma^*)$, whence $\eta(F(\Gamma), 0) = 0$. This means that (counting with multiplicity)

$(*)$ $\qquad \#$(zeros of F enclosed by Γ) - $\#$(poles of F enclosed by Γ) $= 0$.

Now clearly (with all counts taking multiplicties into account):

$$\#(\text{zeros of } F \text{ enclosed by } \Gamma) - \#(\text{poles of } F \text{ enclosed by } \Gamma)$$

is equal to

$$\#(\text{zeros of } g \text{ enclosed by } \Gamma) - \#(\text{zeros of } f \text{ enclosed by } \Gamma).$$

and this gives the result via $(*)$. $\square$

Examples 1.1.1. Here are two applications of Rouché's theorem.

1) Let $P(z) = a_n z^n + \cdots + a_0$, with $a_n \neq 0$. We will show that $P(z)$ has n zeros in $\mathbf{C}$. We can divide by a_n and assume $a_n = 1$. Thus

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0.$$

Let $f(z) = z^n$ and $g(z) = P(z)$. To apply Rouché's Theorem to f and g we wish that $|P(z) - z^n| < |z^n|$ on some circle around the origin. To achieve this, let

$$\rho = \max\left\{1, \left(2\sum_{i=0}^{n-1} |a_i|\right)\right\}.$$

If $|z| > \rho$ then

$$\frac{|f(z) - g(z)|}{|f(z)|} = \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| < \frac{\sum_{i=0}^{n-1} |a_i|}{\rho} < \frac{1}{2}.$$

Thus if $R > \rho$ and D_R is the open disc of radius R centred at 0, and C_R its bounding circle, then applying Rouché on C_R, we see that P has n-roots in D_R. Since $R > \rho$ is arbitrary, we conclude that $P(z)$ has n roots in $\mathbf{C}$.

2) We will show that for each real $\lambda > 1$, the equation $z + e^{-z} = \lambda$ has exactly one solutuon z_0 with $\mathrm{Re}(z_0) > 0$ (i.e., the equation has exactly one solution in the open right half plane). To see this, let $f(z) = z - \lambda$ and $g(z) = z + e^{-z} - \lambda$. For $R > 0$ let γ_R be the contour which is the right semicircle $t \mapsto Re^{it}$ for $-\pi/2 \leq t \leq \pi/2$ followed by the vertical line segment starting at iR and ending at $-iR$. Now $f(z) - g(z) = -e^{-z}$. On the imaginary axis we therefore have (with $y \in \mathbf{R}$)

$$|f(iy) - g(iy)| = |e^{-iy}| = 1 < \sqrt{\lambda^2 + y^2} = |f(iy)|.$$

On the semicircle of points $Re^{i\theta}$ with $-\pi/2 \leq \theta \leq \pi/2$ we have

$$\begin{aligned} |f(Re^{i\theta}) - g(Re^{i\theta})| &= |e^{-Re^{i\theta}}| \\ &= e^{-R\cos\theta} \\ &\leq 1 \quad (\text{since } R\cos\theta > 0 \text{ for } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]) \\ &\leq |Re^{i\theta} - \lambda| \end{aligned}$$

provided $R > \lambda + 1$. But $|Re^{i\theta} - \lambda| = |f(Re^{i\theta})|$. Thus Rouché's theorem's hypotheses are satisfied by f and g on γ_R. Since γ_R encloses only one root of $f(z)$ for $R > \lambda + 1$, therefore so does g. Let $R \to \infty$.

2. **Conformal Maps**

2.1. **Similarity transformation.** Recall an important property of an orthogonal linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$, namely that T is a composition of rotations and reflections, and if $\det T = 1$ it is a rotation. If we are only interested in the angle preserving nature of a transformation, then we are led to the notion of a similarity transformation.

Definition 2.1.1. A linear transformation $L\colon \mathbf{R}^n \to \mathbf{R}^n$ is said to be a *similarity transformation* if there exists a non-zero scalar α such that αL is an orthogonal transformation. A similarity transform is a *conformal linear transformation* if $\det L > 0$.

In practice it is convenient to assume that the scalar α in the definition above is positive. And that will be our implicit assumption from now on.

We can regard the dual of $\mathbf{R}^n$ as equal to $\mathbf{R}^n$ via $\mathbf{e}_i \mapsto \mathbf{e}_i^*$, $i = 1, \dots, n$ where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis of $\mathbf{R}^n$ and $\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$, its dual basis (i.e., $\mathbf{e}_i^*(\mathbf{e}_j) = \delta_{ij}$, $1 \leq i, j \leq n$). Equivalently, the inner product $<,>$ on $\mathbf{R}^n$ allows us to regard each $\boldsymbol{v} \in \mathbf{R}^n$ as a functional (i.e., as an element of the dual of $\mathbf{R}^n$), namely the functional $\boldsymbol{x} \mapsto <\boldsymbol{v}, \boldsymbol{x}>$, and this correspondence is bijective (and linear) between $\mathbf{R}^n$ and its dual. The transpose T^t of a linear operator T on $\mathbf{R}^n$ can therefore be regarded as an operator on $\mathbf{R}^n$ and as such is characterised by the property that $<T\boldsymbol{x}, \boldsymbol{y}> = <\boldsymbol{x}, T^t\boldsymbol{y}>$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbf{R}^n$. With this identification, it is well known that an orthogonal linear transformation T is one such that $T^tT = I_{\mathbf{R}^n}$. In particular, if L is a similarity transformation and $\alpha > 0$ is as above, we have $L^tL = \alpha^{-2}(\alpha L)^t(\alpha L) = \alpha^{-2}I_{\mathbf{R}^n}$. This means , with $\Delta = \det L$, we have $\Delta^2 = \alpha^{-2n}$, i.e.,

$$\alpha = |\Delta|^{-\frac{1}{n}}.$$

Thus another definition of a similarity transformation L on $\mathbf{R}^n$ is that L is non-singular and $L^tL = (\Delta^2)^{1/n} I_{\mathbf{R}^n}$.

Suppose L is a similarity transformation on $\mathbf{R}^n$ and α, Δ are as above. For $\boldsymbol{x}$, $\boldsymbol{y}$ in $\mathbf{R}^n$, it is easy to see that $< L\boldsymbol{x}, L\boldsymbol{y} > = < \boldsymbol{x}, L^tL\boldsymbol{y} > = (\Delta^2)^{1/n} < \boldsymbol{x}, \boldsymbol{y} >$, and from here it is clear that

$$\frac{< L\boldsymbol{x}, L\boldsymbol{y} >}{\|L\boldsymbol{x}\|\|L\boldsymbol{y}\|} = \frac{< \boldsymbol{x}, \boldsymbol{y} >}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|}. \tag{2.1.2}$$

Equation (2.1.2) says that a similarity transformation preserves angles between vectors. But positive scalar multiplications of reflections could occur, so it may not preserve the oriented angles between vectors. However, if $\det L > 0$, i.e., if L is an *conformal linear transform*, then it preserves oriented angles between vectors, since the orthogonal transform αL is then a rotation.

Suppose $n = 2$ and $L\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a conformal linear transformation. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be the matrix corresponding to L with respect to the standard basis on $\mathbf{R}^2$. Since $M^tM = \Delta I_2 = MM^t$ where $\Delta = \det M = ad - bc > 0$, therefore one sees that

(i) $ab + cd = ac + bd = 0$;
(ii) $a^2 + c^2 = b^2 + d^2 = \Delta$.

It is well known, and very easy to see, that this happens if and only if $a = d$ and $b = -c$. Indeed, the above relations between a, b, c, and d show that the the vector $\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$ is a solution of the system

$$\begin{aligned} x^2 + y^2 &= \Delta \\ ax + cy &= 0. \end{aligned} \tag{$*$}$$

There are only two solutions to the system $(*)$ since the solutions are the intersection points of the line $ax+cy = 0$ (which passes through the origin) and the circle centred at the origin of radius $\sqrt{\Delta}$. The two solutions are negatives (antipodes) of each other. Let the two solutions be $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} -x_0 \\ -y_0 \end{smallmatrix}\right)$. Only one of them is such $M = \left(\begin{smallmatrix} a & x \\ c & y \end{smallmatrix}\right)$ has positive determinant (a requirement for conformality), and that solution must be $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} -c \\ a \end{smallmatrix}\right)$ is also a solution of $(*)$ such that $\det \left(\begin{smallmatrix} a & x \\ c & y \end{smallmatrix}\right) = a^2 + c^2 = \Delta > 0$, we see that $\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) = \left(\begin{smallmatrix} -c \\ a \end{smallmatrix}\right)$. In other words, $b = -c$ and $d = a$.

Conversely, if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a non-singular matrix such that $a = d$ and $b = -c$, then clearly $\det M = a^2 + b^2 > 0$ and $M^tM = (a^2 + b^2)I$ whence the linear transformation corresponding to M is conformal.

We therefore have:

Lemma 2.1.3. *A linear transformation $L\colon \mathbf{R}^2 \to \mathbf{R}^2$ is conformal if and only if the matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ corresponding to it (via the standard basis on $\mathbf{R}^2$) satisfies $a = d$, $b = -c$ and* $\det M \neq 0$.

There is another way of viewing this lemma. A conformal linear transformation is a scaled version of a rotation. Rotation by an angle θ in the plane is given by the matrix $R_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. A conformal linear transformation is of the form αR_θ where $\alpha \neq 0$. The Lemma is immediate.

2.2. **Conformal maps on the plane.** We begin with a general definition, even though our interest is only in regions in the plane.

Definition 2.2.1. Let $f\colon U \to \mathbf{R}^n$ be a C^1 map on an open subset U of $\mathbf{R}^n$. The map f is *conformal at* $x \in U$ if the derivative $(Df)(x)\colon \mathbf{R}^n \to \mathbf{R}^n$ is a conformal linear transformation. The map f is *conformal* it is conformal at each point of U.

Remark 2.2.2. If f is conformal at a point x in its domain U, then given any two tangent vectors at x, their oriented angles are preserved by $Df(x)$ in view of the discussion we had above. In other words if γ and σ are two C^1-paths passing through x at time $t = 0$, then the oriented angles between the velocity vectors $d\gamma/dt|_{t=0}$ and $d\sigma/dt|_{t=0}$ is the same as the oriented angle between the tangent vectors $(Df)(x)(d\gamma/dt|_{t=0})$ and $(Df)(x)(d\sigma/dt|_{t=0})$ at $f(p)$. The right way to think of this is that up to a positive scalar multiple, the effect of f on tangent spaces (with the standard Riemannian metric) is that of a rotation. In other words $Df(x)$ takes circles to circles. This is phrased classically as *conformal maps take infinitesimal circles to infinitesimal circles.*

Let us regard linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$ as $m \times n$ matrices in the usual way (via the standard bases on $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively).

Suppose Ω is a region in $\mathbf{C} = \mathbf{R}^2$ and $f\colon \Omega \to \mathbf{C}$ a C^1-map. Let $f = u + iv$ be the decomposition of f into its real and imaginary parts. Then $Df = \left(\begin{smallmatrix} u_x & u_y \\ v_x & v_y \end{smallmatrix}\right)$. From Lemma 2.1.3 we see that f is conformal if and only if $u_x = v_y$, $u_y = -v_x$, and $u_x^2 + u_y^2 \neq 0$. The first two conditions say that f is analytic and the last condition says that $|f'(z)|^2 \neq 0$ for $z \in \Omega$. We thus have,

Proposition 2.2.3. *Let Ω be a region in $\mathbf{C}$. A map $f\colon \Omega \to \mathbf{C}$ is conformal if and only if f is analytic on Ω with nowhere vanishing complex analytic derivative f'.*

Proof. We have already supplied a proof above the statement of the theorem. Here is a proof which is more "function theoretic". Let z_0 be a point in Ω. Let $f\colon \Omega \to \mathbf{C}$ be a C^1-map, with $f = u_i v$ the decomposition of f into its real and imaginary parts. Let $t \mapsto z(t)$ be a C^1-paths with domain a small interval containing 0 as an interior point, say $[-\epsilon, \epsilon]$, such that $z(0) = z_0$ and such that $(dz(t)/dt)|_{t=0} \neq 0$. Let us write $z'(t)$ for $dz(t)/dt$. Let $w(t) = f(z(t))$. Conformality means that $\operatorname{Arg}(\frac{w'(0)}{z'(0)})$ is independent of the choice of the C^1-path $t \mapsto z(t)$ satisfying the hypotheses we have imposed on it[1]. If f is analytic and $f'(z_0) \neq 0$, then it is clear that $(\frac{w'(0)}{z'(0)}) = f'(z_0)$ and we are done. Conversely, suppose $\operatorname{Arg}(\frac{w'(0)}{z'(0)})$ is independent of the path $t \mapsto z(t)$. Now

$$\begin{aligned} w'(t) &= \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) \\ &= \frac{\partial f}{\partial x} \frac{z'(t) + \overline{z'(t)}}{2} + \frac{\partial f}{\partial y} \frac{z'(t) - \overline{z'(t)}}{2i} \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) z'(t) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)\overline{z'(t)} \end{aligned}$$

[1] This is easy to see and left for you to check. Use the fact that if $t \mapsto z_i(t)$, $i = 1, 2$ are two paths through z_0 of the kind we are considering and $t \mapsto w_i(t)$ their images under f then conformality is equivalent to saying $\operatorname{Arg}(\frac{w_2'(0)}{w_1'(0)}) = \operatorname{Arg}(\frac{z_2'(0)}{z_1'(0)})$.

We therefore have.

$$\frac{w'(0)}{z'(0)} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\Bigg|_{z=z_0} + \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)\Bigg|_{z=z_0} \frac{\overline{z'(0)}}{z'(0)}.$$

This can be rewritten as

$$\frac{w'(0)}{z'(0)} = A + Be^{i\theta}$$

where $A = 1/2(f_x(z_0) - if_y(z_0))$, $B = 1/2(f_x(z_0) + if_y(z_0))$, and $e^{i\theta} = z'(0)/\overline{z'(0)}$.

Clearly A and B do not depend on the path $t \mapsto z(t)$, but $e^{i\theta}$ does. The only way $\mathrm{Arg}(\frac{w'(0)}{z'(0)})$ will be independent of θ is if $B = 0$. But the vanishing of B is equivalent to f satisfying the Cauchy-Riemann equations at z_0 and hence f is analytic, since z_0 is an arbitrary point of Ω. □

3. **Laurent Series**

For $a \in \mathbf{C}$ and $0 < r < s$ let $A(a, r, s)$ be the annulus around a between the circle of radius r and the circle of radius s, both centred at a. In other words

$$A(a, r, s) = \{z \in \mathbf{C} \mid r < |z - a| < s\}.$$

Let C_1 and C_2 be the (oriented) circles of radius r and s centred at a. Suppose $f(z)$ is holomorphic on $\overline{A}(a, r, s)$, the closed annulus bounded by C_1 and C_2. Let $\Gamma = C_2 - C_1$. It is clear that Γ is homologous to zero in the domain of $f(z)$, for if c does not lie in the domain of $f(z)$ then c does not lie in $\overline{A}(a, r, s)$, whence, either $|c - a| < r$ or $|c - a| > s$. In either case it is trivial to see that $\eta(\Gamma, c) = 0$. Finally if $c \in A(a, r, s)$ then $\eta(\Gamma,\, c) = \Gamma(C_2,\, c) = 1$. The generalised Cauchy-Goursat theorem then gives

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta \quad (z \in A(a, r, s)).$$

Consequently

$$f(z) = f_1(z) + f_2(z) \quad (z \in A(a, r, s))$$

where, for z in the annular region we are considering

$$f_1(z) = -\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}d\zeta$$

and

$$f_2(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}d\zeta.$$

From the integral formulas above, it is clear that $f_1(z)$ is holomorphic on the region $|z - a| > r$ and $f_2(z)$ is holomorphic on the region $|z - a| < s$. In fact, a little thought shows that if ρ is such that $r < \rho < s$ and C_ρ is the circle with centre a and radius ρ, then for $|z - a| > \rho$ we have $f_1(z) = -\frac{1}{2\pi i}\int_{C_\rho} \frac{f(\zeta)}{\zeta - z}d\zeta$ and for $|z - a| < \rho$ $f_2(z) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(\zeta)}{\zeta - z}d\zeta$. Now, from earlier results of this course $f_2(z)$ has a power series expansion in around a in the disc $|z - a| < s$,

$$f_2(z) = \sum_{n=0}^{\infty} A_n(z - a)^n$$

where the coefficients A_n are given by

$$(*) \qquad A_n = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta \qquad (n \geq 0).$$

As for $f_1(z)$ we have for $|z - a| > r$

$$\begin{aligned} f_1(z) &= -\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= -\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta - a) - (z - a)} d\zeta \\ &= -\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(z - a)(\frac{\zeta - a}{z - a} - 1)} d\zeta \\ &= \frac{1}{2\pi i}\int_{C_1} \sum_{n=0}^{\infty} \frac{(\zeta - a)^n}{(z - a)^{n+1}} d\zeta \\ &= \frac{1}{2\pi i}\int_{C_1} \sum_{n=1}^{\infty} \frac{(\zeta - a)^{n-1}}{(z - a)^n} d\zeta. \end{aligned}$$

Standard uniform convergence arguments (check this!) allowing the sum and the integral to be interchanged gives

$$f_1(z) = \sum_{n=1}^{\infty} B_n (z - a)^{-n}$$

where for $n \geq 1$, $B_n = \frac{1}{2\pi i}\int_{C_1} f(\zeta)(\zeta - a)^{n-1} d\zeta$. This last integral is the same if taken over C_ρ by Stokes' Theorem or by the genaralised Cauchy-Goursat theorem. We thus have

$$(**) \qquad B_n = \frac{1}{2\pi i}\int_{C_\rho} f(\zeta)(\zeta - a)^{n-1} d\zeta \qquad (n \geq 1).$$

Putting together $(*)$ and $(**)$ we see that:

Theorem 3.1. *Let $f(z)$ be holomorphic on the annulus*

$$A\{z \in \mathbf{C} \mid r < |z - a| < s\}$$

centred at a, where $r = 0$ and $s = \infty$ are allowed. The function $f(z)$ can be expressed as a series

$$(3.1.1) \qquad \begin{aligned} f(z) &= \cdots + \frac{a_{-m}}{(z - a)^m} + \cdots + \frac{a_{-1}}{z - a} \\ &\quad + a_0 + a_1(z - a) + \cdots + a_n(z - a)^n + \dots \\ &= \sum_{n=-\infty}^{\infty} a_n (z - a)^n \end{aligned}$$

which converges uniformly on compact subsets of A. The coefficients a_n are given by the integral formula

$$(3.1.2) \qquad a_n = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta \qquad (n \in \mathbb{Z})$$

where $\rho \in (r, s)$ and C_ρ is the circle of radius ρ centred at a. The series

$$f_1(z) = \sum_{n \leq -1} a_n (z-a)^n$$

converges uniformly on compact sets in the unbounded region $|z-a| > r$ and the series

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

converges uniformly on compact sets in the disc $|z-a| < s$.

Proof. Our discussion before the statement was for bounded annuli where the inner radius $r \neq 0$ and $s \neq \infty$ and f was holomorphic on the closure of the annulus under study. We are now dealing with $f$holomorphic on the open annulus A, and allowing r to equal zero, and s to equal ∞. Pick r' and s' satisfying $r < r' < s' < s$ and we see that the statement of the theorem is valid on $A(a, r', s')$ based on our discussion before the statement (see $(*)$ and $(**)$ for formulas for the coefficients a_n). Since r' and s' are arbitrary in the range $r < r' < s' < s$, we are done. $\square$

Definition 3.1.3. For $f(z)$ as in Theorem 3.1, the series (3.1.1) is called the *Laurent series of $f(z)$ centred at a.* The series in the negative powers of $z-a$, i.e., $f_1(z) = \sum_{n \leq -1} a_n/(z-a)^n$ is called the *principal part* of the Laurent expansion of $f(z)$ around a.

3.2. **Residues.** Suppose $f(z)$ has an isolated singularity at some point $a \in \mathbf{C}$. According to Theorem 3.1 we have a Laurent series expansion of $f(z)$ around a,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n.$$

This expansion is valid in any punctured disc $B^* = B^*(a, r) = B(a, r) \smallsetminus \{a\}$ on which $f(z)$ is analytic.

Definition 3.2.1. With $f(z)$ and a_n, $n \in \mathbb{Z}$, as above, the residue of f at a, denoted $\mathrm{Res}_{z=a} f(z)$ is

$$\mathrm{Res}_{z=a} f(z) = a_{-1}.$$

Let γ be a closed path in B^* such that $\eta(\gamma, a) = 1$. Note that if $n \neq -1$, the function $(z-a)^n$ has a primitive in B^* and hence $\int_\gamma (z-a)^n dz = 0$ for all such n. Since the convergence of the Laurent series is uniform on compact subsets of B^*, we have, from the observation we just made

$$\frac{1}{2\pi i} \int_\gamma f(z) dz = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_\gamma (z-a)^n dz = \frac{1}{2\pi i} \int_\gamma \frac{a_{-1}}{z-a} dz = \mathrm{Res}_{z=a} f(z).$$

We thus have

Lemma 3.2.2. *Let $f(z)$, B^* and γ be as above.*

(a) *The residue of $f(z)$ at a is given by the formula $\mathrm{Res}_{z=a} f(z) = (2\pi i)^{-1} \int_\gamma f(z) dz$.*
(b) *The residue of $f(z)$ at a is characterised by the property that it the only complex number R such that $f(z) - R(z-a)^{-1}$ has a primitive in B^*.*

Proof. We have already seen part (a). Part (b) is an immediate consequence, since $\int_\gamma f(z)-R(z-a)^{-1}dz$ has to equal zero if $f(z)-R(z-a)^{-1}$ has a primitive, giving $R = \mathrm{Res}_{z=a} f(z)$. Conversely, if $R = \mathrm{Res}_{z=a} f(z)$, then as we observed earlier, $f(z) - R(z-a)^{-1}$ has a primitive, since its Laurent expansion is such that the coefficient of $(z-a)^{-1}$ is zero. □

4. **Families of holomorphic functions**

4.1. **Locally bounded families.** Let Ω be a region and $\mathscr{F}$ a non-empty family of holomorphic functions on Ω.

Definition 4.1.1. $\mathscr{F}$ is a said to be *locally bounded* if for each compact subset K of Ω, there exists M_K such that for every $z \in K$ and every $f \in \mathscr{F}$ we have $|f(z)| < M_K$.

Closely related (in fact, via a theorem of Paul Montel, equivalent) to the notion of locally bounded families is the notion of uniformly equicontinuous families.

Definition 4.1.2. $\mathscr{F}$ is said to be *uniformly equicontinuous on compact sets (UECS)* if for each compact subset K of Ω and $\epsilon > 0$, there exists $\delta = \delta_K > 0$ such that $|f(z) - f(w)| < \epsilon$ whenever $z, w \in K$ and $|z - w| < \delta$.

Recall that the Arzela-Ascoli theorem says that if $\mathscr{F}$ is a family of continuous functions on a compact subset K of $\mathbf{C}$ then it is totally bounded (i.e., its closure is compact) if and only if it is uniformly bounded and equicontinuous. The next result says that for a family of holomorphic functions, the notion of equicontinuity can be dropped. More precisely we have:

Lemma 4.1.3. *A locally bounded family of holomorphic functions on Ω is UECS.*

Proof. Let $\mathscr{F}$ be locally bounded. Let $K = \overline{B}(z_0, \rho)$ be a closed ball contained in Ω. By definition of a locally bounded family we can find a positive real number M_K such that $|f(z)| < M_K$ for all $z \in K$. Since the distance between K and $\mathbf{C} \setminus \Omega$ is strictly greater than 0, we can find $r > \rho$ such that $\overline{B}(z_0, r) \subset \Omega$. Let $\epsilon > 0$ be given. Set

$$\delta_K = \frac{\epsilon(r-\rho)^2}{M_K r}.$$

Then for $z, w \in K$ with and $f \in \mathscr{F}$ we have (with C_r the circle with radius r centred at z_0)

$$\begin{aligned}
|f(z) - f(w)| &= \frac{1}{2\pi}\left|\int_{C_r}\left[\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w}\right]d\zeta\right| \\
&= \frac{1}{2\pi}\left|\int_{C_r}\frac{(z-w)f(\zeta)}{(\zeta - z)(\zeta - w)}d\zeta\right| \\
&\le \frac{|z-w|M_K}{2\pi}\int_{C_r}\left|\frac{1}{(\zeta - z)(\zeta - w)}\right||d\zeta| \\
&\le \left(\frac{M_K}{2\pi}\frac{1}{(r-\rho)^2}2\pi r\right)|z-w| \\
&= \frac{rM_K|z-w|}{(r-\rho)^2}.
\end{aligned}$$

This last quantity is less than ϵ if $|z - w| < \delta_K$ by our choice of δ_K above. $\square$