LECTURE 12 AND 13

Dates of Lectures: February 15 and 16, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

1. **Closed and exact forms on regions in C**

1.1. Our conventions and assumptions are as follows. By a *form* on an open set U of $\mathbf{C}$ we mean a C^1 1-form on U. Such a form ω can be expressed in terms of the real and imaginary coordinates x and y as

$$\omega = pdx + qdy$$

where $p, q\colon U \to \mathbf{C}$ are C^1 functions. We have the exterior derivative d on n-forms. Recall that for a function f in U with partial derivatives the exterior derivative is given by $df = f_x dx + f_y dy$ and for a 1-form $\omega = pdx + qdy$ on U with p and q possessing partial derivatives (not necessarily continuous, i.e., ω is not necessarily a form), $d\omega = (q_x - p_y)dx \wedge dy$.

A form ω is said to be closed if $d\omega = 0$ and it is said be exact if $\omega = df$ for some C^2 function f on Ω.

Note that if $\gamma\colon [a,\, b] \to U$ is closed path in U and ω is an exact form then the line integral $\int_\gamma \omega = 0$. Indeed, suppose $\omega = df$ where f is C^2, then

$$\int_\gamma \omega = \int_\gamma df = \int_{[a,b]} \gamma^*(df) = \int_{[a,b]} d(f\circ\gamma) = f(\gamma(b)) - f(\gamma(a)) = 0.$$

It is clear that the closed path γ in the above formula can be replaced by a cycle Γ to give $\int_\Gamma \omega = 0$. We need the following (a version of the Poincaré Theorem)

Lemma 1.1.1. *Let $B = B(a,r)$ be an open ball in $\mathbf{C}$ and ω a closed form on B. Then ω is exact. In particular*

$$\int_\Gamma \omega = 0$$

for all cycles Γ in B.

Proof. Let $\omega = pdx + qdy$. For $z \in B$ let L_z be line segment starting at a and ending at z. Define

$$u(z) = \int_{L_z} \omega.$$

If $z = x + iy$ and $a = c + id$ (x, y, c, d real numbers) then we have four paths, $\sigma = [a, x + id]$, $\tau = [x + id, z$, $\tau' = [a, c + iy]$ and $\sigma' = [c + iy, z]$. Since ω is closed, by Stokes' Theorem $u(z) = \int_\sigma \omega + \int_\tau \omega = \int_{\sigma'} \omega + \int_{\tau'} \omega$. From the fundamental theorem of calculus it is not hard to see that $u_x = p$ and $u_y = q$. Hence $\omega = du$ whence it is exact. $\square$

2. Homology theory for cycles

2.1. Cycles homologous to zero. In complex analysis, the following definition is used for cycles homologous to zero in a region.

Definition 2.1.1. Let Ω be a region. A cycle Γ in Ω is said to be homologous to zero in Ω if $\eta(\Gamma, a) = 0$ for every $a \notin \Omega$.

If Γ is homologous to zero in Ω, we often write $\Gamma \sim 0 \pmod{\Omega}$. If Γ and Γ' two cycles in Ω and $\Gamma - \Gamma' \sim 0 \pmod{\Omega}$, we often write $\Gamma \sim \Gamma' \pmod{\Omega}$ and say Γ is homologous to Γ' with respect to Ω. Note that if $\Gamma \sim 0 \pmod{\Omega}$ then $\Gamma \sim 0$ (mod Ω') for every $\Omega' \supset \Omega$.

2.2. Grid cycles. A cycle Σ in the complex plane is said to be a *grid cycle* if it is of the form

$$\Sigma = \sum_{i=1}^{m} n_i \sigma_i$$

where $n_i \in \mathbb{Z}$ and σ_i is either a horizontal segment of the form $[z, z+h$ where h is a positive real number or of the form $[z, z+ik]$ where k is a positive real number.

Lemma 2.2.1. *Let Ω be a region and Γ a cycle in Ω. Then there exists a grid cycle Γ_0 in Ω such that*

$$\int_\Gamma \omega = \int_{\Gamma_0} \omega$$

for every closed form ω on Ω. (In particular, taking $\omega = dz/(z-a)$ for $a \notin \Omega$ we have $\Gamma \sim \Gamma_0$ (mod Ω).)

Proof. Without loss of generality we may assume Γ is a closed path $\gamma\colon [a, b] \to \Omega$. Let $M > 0$ be a real number such that the distance between the compact set γ^* and $\mathbf{C} \smallsetminus \Omega$ is $< M$. By uniform continuity of γ ($[a, b]$ is compact!) there exists $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < M$ whenever $|s - t| < \delta$. Taking a partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that the length of each sub-interval determined by successive point in the partition is less that δ we see that each $\gamma([t_{i-1}, t_i])$, $i = 1, \ldots, n$ is contained in an open ball B_i of radius $< M$ and centered at (say) $\gamma((t_{i-1} + t_i)/2)$. It follows that B_i is contained in Ω. Moreover we can connect $\gamma(t_{i-1})$ with $\gamma(t_i)$ in B_i by two line segments, one of which is horizontal and the other vertical, and let σ_i be such a chain. Let $\gamma_i = \gamma|_{[t_{i-1},\, t_i]}$. Then from Lemma 1.1.1 we have $\int_{\gamma_i} \omega = \int_{\sigma_i} \omega$ for every closed form ω. Set $\Gamma_0 = \sum_{i=1}^{n} \sigma_i$. It is clear that Γ_0 is a cycle and that $\int_\gamma \omega = \int_{\Gamma_0} \omega$ for every closed form ω on Ω. $\square$

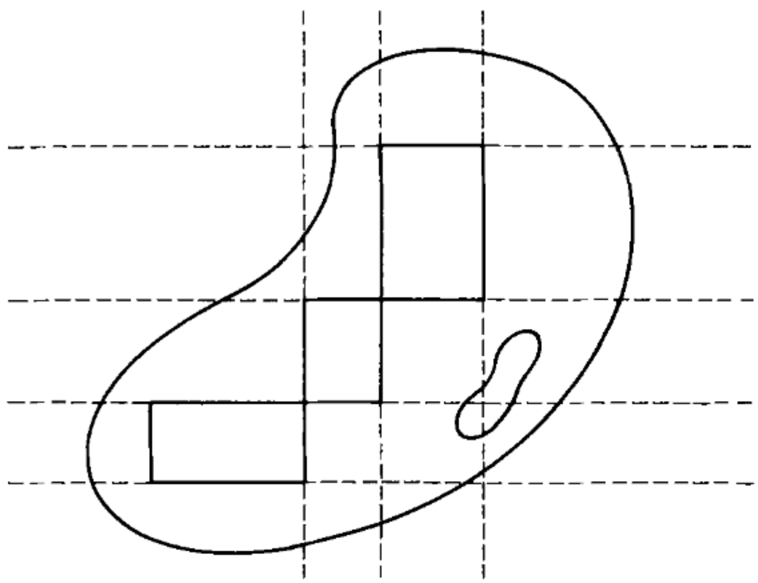
2.3. Closed forms and cycles homologous to zero. The proof of the main theorem below is due to E. Artin. See [A, § 4.6, pp.144–146].

Theorem 2.3.1. *Let Ω be a region and Γ a cycle in Ω such that $\Gamma \sim 0$ (mod Ω). Then*

$$\int_\Gamma \omega = 0$$

for every closed form ω.

Proof. In view of Lemma 2.2.1 we may assume Γ is a grid cycle. Then Γ is the sum of integral multiple of finite length line segments in Ω which are either horizontal or vertical. Suppose σ is such a segment. Consider the infinite line supporting σ^* and take the union all lines so obtained. This gives a grid in $\mathbf{C}$ with constituent closed rectangles being finite in number (since the number of horizontal and vertical lines constituting the grid is finite in number). Some of these rectangles are bounded and others are infinite. See the picture below from [A, Fig.4-11, p.145]



Let R_i, $i = 1, \dots, n$ be the bounded rectangles and Q_r, $r = 1, \dots, m$ the infinite rectangles. Pick points a_i (resp. q_r) in the interior of R_i (resp. Q_r) for each $i \in \{1, \dots, n\}$) (resp. $r \in \{1, \dots, m\}$). Define a new cycle in $\mathbf{C}$

$$\Sigma = \sum_{i=1}^{n} \eta(\Gamma,\, a_i)\partial R_i. \tag{2.3.1.1}$$

The convention we use is that ∂R for any finite closed rectangle its boundary ∂R is a grid cycle with the oriented edges being the constituent segments, the orientation of an edge being such that R is to its left when one moves along the oriented segment in the direction of the orientation. It is easy to see that

$$\eta(\Gamma, a) = \eta(\Sigma, a) \qquad (a \notin \Gamma^* \cup \Sigma^*). \tag{†}$$

For a chain Δ in $\mathbf{C}$ let $\overline{\Delta}$ denote its class in $C(\mathbf{C})/\mathfrak{R}(\mathbf{C})$. We claim that $\Gamma - \Sigma \in \mathfrak{R}(\mathbf{C})$. To see this note that

$$\overline{\Gamma} - \overline{\Sigma} = \sum_{\overline{\sigma}} d_{\overline{\sigma}}\overline{\sigma} \tag{$*$}$$

where $d_{\overline{\sigma}} \in \mathbb{Z}$ and $\overline{\sigma}$ runs through $\mathfrak{R}(\mathbf{C})$ equivalence classes of oriented edges of ∂R_i, $i \in \{1, \dots, n\}$. We can write the expression in $(*)$ so that it is a *reduced* expression, i.e., each index $\overline{\sigma}$ occurs only once as an index. Suppose σ is a constituent edge of ∂R where R is one of $R_1, \dots, R_n$. Then either σ^* is a common (un-oriented) edge of two finite rectangles or the common edge of a finite rectangle and an infinite rectangle. In the first case, suppose R_i and R_j have σ^* as a common un-oriented edge, with R_i lying to the left of σ and R_j to the right. Consider the cycle class

$$\overline{\Delta} = \overline{\Gamma} - \overline{\Sigma} - d_{\overline{\sigma}}\overline{\partial R_i}.$$

The coefficient of $\overline{\sigma}$ in this cycle is zero. It follows that $\eta(\Delta, a_i) = \eta(\Delta, a_j)$, since the common boundary between R_i and R_j does not figure in the expression for Δ modulo $\mathfrak{R}(\mathbf{C})$. Using (†) it follows that

$$\eta(-d_{\overline{\sigma}}\partial R_i,\, a_i) = \eta(-d_{\overline{\sigma}}\partial R_i,\, a_j).$$

This means $d_{\overline{\sigma}} = 0$. Now suppose σ^* is the common edge of a finite rectangle R_i and an infinite rectangle Q_r. We may assume that σ is so oriented that R_i is to the left of σ (otherwise use $-\sigma$, and make the corresponding change in $(*)$). Again using (†) and the same argument as before (with Q_r replacing R_j) we see that $d_{\overline{\sigma}} = 0$ in this case too. Thus from $(*)$ we have $\overline{\Gamma} = \overline{\Sigma}$.

We claim that Σ is a cycle in Ω. This means, from the definition of Σ in (2.3.1.1), we have to show that if $\eta(\Gamma,\, a_i) \neq 0$ then $R_i \subset \Omega$. The contrapositive lends itself to a more natural proof. If R_i has a point a such that $a \notin \Omega$ then $\eta(\Gamma,\, a) = 0$ for $\Gamma \sim 0 \pmod{\Omega}$. One can connect a_i to a be a line segment which avoids Γ. Thus $\eta(\Gamma,\, a_i) = 0$. This proves the contrapositive of the statement we wished to prove.

Since Σ is a cycle in Ω, it makes sense to integrate forms on Ω over Σ and since $\overline{\Sigma} = \overline{\Gamma}$, $\int_\Sigma \omega = \int_\Gamma \omega$ for all forms ω on Ω. If ω is closed, we have

$$\int_\Sigma \omega = \sum_{i=1}^{n} \eta(\Gamma,\, a_i) \int_{\partial R_i} \omega = \sum_{i=1}^{n} \eta(\Gamma,\, a_i) \iint_{R_i} d\omega = 0.$$

This proves the result. □

Corollary 2.3.2. *A region Ω is simply connected if and only if for every cycle Γ and every closed form ω*

$$\int_\Gamma \omega = 0. \tag{2.3.2.1}$$

Proof. According to Theorem 2.1.2 of Lecture 11 a region is simply connected if and only if every cycle Γ in Ω is homologous to zero in Ω.

Suppose (2.3.2.1) holds for every pair (Γ, ω) with Γ a cycle and ω a closed form. Since $dz/(z-a)$ is a closed form on Ω for every $a \notin \Omega$, we get $\eta(\Gamma, a) = 0$ for every cycle Γ and every $a \notin \Omega$. By Theorem 2.1.2 of Lecture 11, Ω is simply connected.

Conversely if Ω is simply connected, then according to Theorem 2.1.2 of Lecture 11 every cycle Γ in Ω is homologous to zero in Ω, and so by Theorem 2.3.1, (2.3.2.1) holds for every pair (Γ, ω) with Γ a cycle and ω a closed form. □

3. The General Form of the Cauchy-Goursat Theorem

3.1. Theorem 2.3.1 has as a corollary the definitive form of the Cauchy-Goursat Theorem, which has further corollaries.

Theorem 3.1.1 (The General Form of the Cauchy-Goursat Theorem)**.** *Let Ω be a region and Γ a cycle such that $\Gamma \sim 0 \pmod{\Omega}$. Then*

$$\int_\Gamma f(z)dz = 0$$

for all holomorphic functions $f(z)$ on Ω.

Proof. By the fact that a holomorphic function $f(z)$ on Ω is infinitely differentiable (and so certainly C^1) and satisfies the Cauchy-Riemann equations we see that $f(z)dz$ is a closed form on Ω. Theorem 2.3.1 gives the rest. $\square$

Corollary 3.1.2 (General form of the Cauchy Integral Formula)**.** *Suppose* Γ *is a cycle in a region* Ω *with* $\Gamma \sim 0$ *(mod* Ω*) and* $f(z)$ *is holomorphic on* Ω*. Then*

$$\eta(\Gamma, z)f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta \qquad (z \in \Omega \smallsetminus \Gamma^*).$$

Proof. Follows easily from Theorem 3.1.1. Note that for fixed $z \in \Omega \smallsetminus \Gamma^*$ the function

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \text{if } \zeta \in \Omega \smallsetminus \{z\} \\ \\ f'(z) & \text{if } \zeta = z. \end{cases}$$

is holomorphic on Ω by Riemann's removable singularities theorem and hence its integral over Γ is zero by Theorem 3.1.1. $\square$

3.2. Simple connectedness. Here is a list of characterisations of simple-connectedness.

Theorem 3.2.1. *Let* Ω *be a region. The following are equivalent*

(1) Ω *is simply connected.*
(2) *Every cycle in* Ω *is homologous to zero in* Ω*.*
(3) *For every cycle* Γ *in* Ω *and every closed form* ω *on* Ω *we have* $\int_\Gamma w = 0$*.*
(4) *For every closed path* γ *in* Ω *and every holomorphic function* $f(z)$ *on* Ω*, we have* $\int_\gamma f(z)dz = 0$*.*
(5) *Every holomorphic function on* Ω *has a primitive.*

Proof. From Theorem 2.1.2 of Lecture 11 and from Corollary 2.3.2 above we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Moreover it is clear that (4) $\Leftrightarrow$ (5) (see, if you need to, Theorem 1.1 of Lecture 3). Next, from Cauchy-Riemann equations, we know that $f(z)dz$ is closed on Ω whenever $f(z)$ is holomorphic on Ω, which yields (3) $\Rightarrow$ (4). Finally (4) $\Rightarrow$ (2) is seen as follows. Suppose $a \notin \Omega$. Then $f(z) = 1/(z - a)$ is a holomorphic function on Ω and hence by (4) we get $\eta(\gamma, a) = 0$ for every closed path in Ω, i.e., $\eta(\Gamma, a) = 0$ every cycle Γ in Ω. $\square$

3.3. The Argument Principle. The following is an important variant of the argument principle.

Theorem 3.3.1. *Suppose* f *is a non-zero meromorphic function on a region* Ω *with zeros at* a_i*,* $i \in I$ *and poles at* b_j*,* $j \in J$*. Suppose the order of the zero at* a_i *is* n_i *for every* $i \in I$ *and the order of the pole at* b_j *is* m_j *for every* $j \in J$*. Then for a cycle* Γ *in* Ω *such that* $\Gamma \sim 0$ *(mod* Ω*) and such that* Γ^* *does not contain any zero or pole of* $f(z)$ *we have*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = \sum_{i\in I}\eta(\Gamma, a_i)n_i - \sum_{j\in J}\eta(\Gamma, b_j)m_j.$$

Proof. The case where f is constant is trivial. So we assume f is non-constant. Let $T = \{a_i \mid i \in I\} \cup \{b_j \mid j \in J\}$ and suppose $U = \{a \in \mathbf{C} \smallsetminus \Gamma^* \mid \eta(\Gamma, a) = 0\}$. Then U is an open set in $\mathbf{C} \smallsetminus \Gamma^*$ (for winding numbers are locally constant). Moreover U contains the unbounded component of $\mathbf{C} \smallsetminus \Gamma^*$ as well as $\mathbf{C} \smallsetminus \Omega$, for $\Gamma \sim 0$ (mod

Ω). The fact that U contains the unbounded component of $\mathbf{C} \smallsetminus \Gamma^*$ shows that $K = \mathbf{C} \smallsetminus U$ is compact and the fact that $U \supset \mathbf{C} \setminus \Omega$ means $K \subset \Omega$. We point out that $K = \{a \in \mathbf{C} \smallsetminus \Gamma^* \mid \eta(\Gamma, a) \neq 0\} \cup \Gamma^*$.

Break up the discrete subset T of Ω as a disjoint union

$$T = S \sqcup F$$

with $S = U \cap T$ and $F = K \cap T$. Since F is discrete in K, it is finite (hence the choice of the symbol F to denote it). Let us reindex if necessary and suppose that $\{a_i \mid i \in I\} \cap F = \{a_1, \ldots, a_r\}$ and $\{b_j \mid j \in J\} \cap F = \{b_1, \ldots, b_s\}$.

Since $S \cap K = \emptyset$ with S discrete in Ω and K compact, we can find an open neighbourhood Ω' of K in Ω such that $\Omega' \cap S = \emptyset$. Note that $\Gamma \sim 0 \pmod{\Omega'}$ for if $\eta(\Gamma, a) \neq 0$ then $a \in K \subset \Omega'$. Clearly Ω' contains Γ^* as well as F. On Ω' the set F is the set of zeros as well as poles of $f(z)$ and hence on Ω' we can write

$$f(z) = g(z) \frac{\prod_{i=1}^{r} (z - a_i)^{n_i}}{\prod_{j=1}^{s} (z - b_j)^{m_j}}$$

with g holomorphic and nowhere vanishing on Ω'. We therefore have (on Ω')

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{i=1}^{r} \frac{n_i}{z - a_i} - \sum_{j=1}^{s} \frac{m_j}{z - b_j}$$

and since $g'(z)/g(z)$ is holomorphic on Ω' and $\Gamma \sim 0 \pmod{\Omega'}$ the integral of $g'(z)/g(z)$ over Γ vanishes by the general form of the Cauchy-Goursat theorem. The result follows easily from this. $\square$

References

[A] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New-York, 1979.