LECTURE 11

Date of Lecture: February 9, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Others are a quick summary.

We ended Lecture 10 with the statement that Ω is simply connected (in the sense of function theory) if and only if $\eta(\gamma, a) = 0$ for every closed path γ in Ω and every $a \notin \Omega$. We deferred the proof to this lecture. In fact we prove a seemingly more general statement in Theorem 2.1.2 below.

1. Chains and cycles

1.1. **Chains.** Let Ω be a region. A chain in Ω is an element of the *free abelian group* $C(\Omega)$ generated by paths in Ω.

$$C(\Omega) = \bigoplus_{\gamma} \mathbb{Z} \cdot \gamma$$

For $\Gamma = \sum_i n_i \gamma_i$ in $C(\Omega)$, define

$$\Gamma^* = \cup_i \gamma_i^*.$$

It is clear that given a chain $\Gamma = \sum_i n_i \gamma_i$, the integral

$$\int_\Gamma f(z)dz = \sum_i n_i \int_{\gamma_i} f(z)dz$$

makes sense for continuous functions f on Γ^*. We call Γ^* the *support* of Γ.

We are interested in an equivalence relation between chains so that equivalent chains have the same integral for all continuous functions in the union of their supports. Let $\mathfrak{R} = \mathfrak{R}(\Omega)$ be the subgroup of $C(\Omega)$ generated by the following relations (see [A, § 4.1, pp.137–138]):

(1) If $\gamma\colon [a, b] \to \Omega$ is a path, and $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of $[a, b]$, and γ_i the restriction of γ to $[t_{i-1}, t_i]$ for $i = 1, \ldots, n$, then

$$\gamma - \sum_{i=1}^{n} \gamma_i \in \mathfrak{R}(\Omega).$$

(2) If γ_i, $i = 1, \ldots, n$ are paths such that the end point b_i of γ_i coincides with the initial point a_{i+1} of γ_{i+1} for $i = 1, \ldots, n-1$, and $\gamma_1 * \cdots * \gamma_n$ is their fusion, then

$$\gamma_1 * \cdots * \gamma_n - \sum_{i+1}^{n} \gamma_i \in \mathfrak{R}(\Omega).$$

(3) If $\varphi\colon [\alpha, \beta] \to [a, b]$ is a C^1 one-to-one map, with $\varphi(s) \neq 0$ for any $s \in [\alpha, \beta]$, and $\gamma\colon [a, b] \to \Omega$ is a path, then

$$\gamma - \gamma \circ \varphi \in \mathfrak{R}(\Omega).$$

In other words, a path and a re-parameterisation of the path differ by an element in $\mathfrak{R}(\Omega)$.

(4) If $\gamma\colon [a,\, b] \to \Omega$ is a path, and $\sigma\colon [-b,\, -a] \to \Omega$ is the *opposite path*, namely $\sigma(t) = \gamma(-t)$ for $t \in [-b,\, -a]$, then

$$\gamma + \sigma \in \mathfrak{R}(\Omega).$$

We point out that

$$\int_\Gamma f(z)dz = 0$$

for every $\Gamma \in \mathfrak{R}(\Omega)$ and every continuous function f on Γ^*, whence

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz \tag{1.1.1}$$

for chains Γ_1 and Γ_2 such that $\Gamma_1 - \Gamma_2 \in \mathfrak{R}(\Omega)$ and functions f which are continuous on $\Gamma_1^* \cup \Gamma_2^*$.

1.2. **Cycles.** A chain $\Gamma = \sum_i n_i\gamma_i$ in Ω is said the be *effective* if all the n_i are non-negative. If Γ is an effective non-zero chain, it can written uniquely as the sum of paths (with repetitions allowed to take care of multiplicites)

$$\Gamma = \gamma_1 + \cdots + \gamma_n. \tag{$*$}$$

If $[a_, b_i]$ is the domain of γ_i in the representation $(*)$ of a non-zero effective chain Γ, then we can form a sequence of initial points $I(\Gamma) = (\gamma_i(a_i),\, \gamma_2(a_2), \ldots,\, \gamma_n(a_n))$ and a sequence of final points $F(\Gamma) = (\gamma_i(b_i),\, \gamma_2(b_2), \ldots,\, \gamma_n(b_n))$. An effective chain Γ in Ω is said to be a *cycle in* Ω if it is either zero or if $I(\Gamma)$ and $F(\Gamma)$ are the same up to permutation. It is easy to see that an effective chain in Ω is a cycle in Ω if and only if it is $\mathfrak{R}(\Omega)$-equivalent to a sum of closed paths in Ω.

If Γ is a chain and every path with a negative coefficient is replaced by its opposite path, then we obtain an effective chain $\widetilde{\Gamma}$. Note that $\Gamma - \widetilde{\Gamma} \in \mathfrak{R}(\Omega)$.

Definition 1.2.1. A chain Γ in Ω is a *cycle* in Ω if the effective chain $\widetilde{\Gamma}$ is a cycle.

Clearly every element of $\mathfrak{R}(\Omega)$ is a cycle in Ω.

From earlier comments, it is clear that Γ is a cycle in Ω if and only if it is $\mathfrak{R}(\Omega)$-equivalent to an integral linear combination of closed paths in Ω.

Let $Z(\Omega) \subset C(\Omega)$ denote the subset of cycles. It is clear that $Z(\Omega)$ is a subgroup containing $\mathfrak{R}(\Omega)$. We thus have a chain of subgroups

$$\mathfrak{R}(\Omega) \subset Z(\Omega) \subset C(\Omega).$$

1.3. **The winding number of a cycle class.** We define the *winding number* of a cycle Γ around a point $a \notin \Gamma^*$ by the formula:

$$\eta(\Gamma,\, a) := \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a}.$$

Clearly $\eta(\Gamma,\, a) \in \mathbb{Z}$. In greater generality, if $\overline{\Gamma}$ is a cycle class modulo $\mathfrak{R}(\mathbf{C})$ and $a \in \mathbf{C}$ a point such that for some representative Γ of $\overline{\Gamma}$ we have $a \notin \Gamma^*$, then the integer $\eta(\Gamma,\, a)$ does not depend on the chosen representative with that property, and we have a well-defined winding number

$$\eta(\overline{\Gamma},\, a) \in \mathbb{Z}$$

for such $a \in \mathbf{C}$. These observations are easy to verify.

The following is trivial following easily from the analogous statement for closed paths.

Proposition 1.3.1. *Let Γ be a cycle. Then*

$$\eta(\Gamma, a) = 0$$

for all points a lying in the unbounded component of $\mathbf{C} \setminus \Gamma^*$.

2. **Simply connected regions**

2.1. The definition used by function theorists is the one in Definition 2.1.1 below. We denote the Riemann sphere by $\widehat{\mathbf{C}}$.

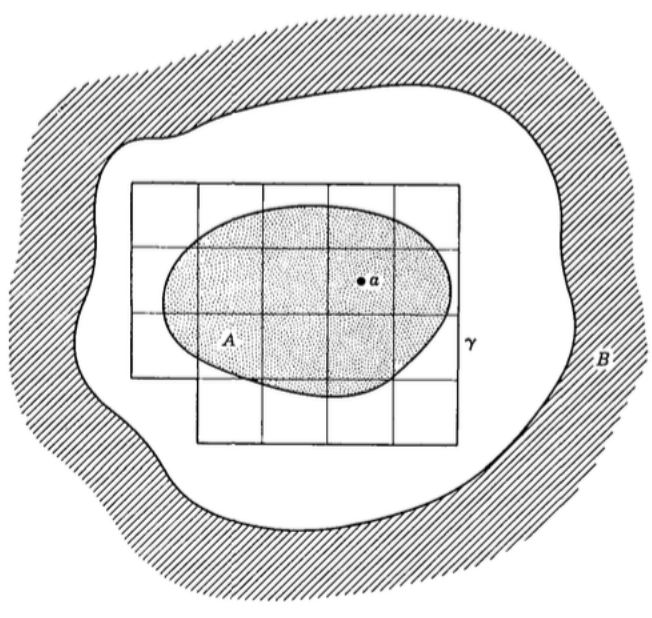
Definition 2.1.1. A region Ω in $\mathbf{C}$ is *simply connected* if $\widehat{\mathbf{C}} \smallsetminus \Omega$ is connected.

Recall that in Lecture 10 ended with a statement which is clearly equivalent to:

Theorem 2.1.2. *A region Ω is simply connected if and only if $\eta(\Gamma, a) = 0$ for all cycles Γ in Ω and all points $a \in \mathbf{C} \smallsetminus \Omega$.*

Proof. Proposition 1.3.1 gives the necessity of the condition.

For the sufficiency of the condition, let $\mathbf{C} \smallsetminus \Omega = A \cup B$, with B the unbounded connected component of $\mathbf{C} \smallsetminus \Omega$, and $A \cap B = \emptyset$. Suppose Ω is not simply connected then neither A nor B is empty. The set A is therefore non-empty and compact. Let $\delta > 0$ be the shortest distance between A and B. Pick a point $a \in A$. Cover the whole plane with a net of closed squares[1] of side $\delta/\sqrt{2}$ such that a is in the centre of some square in out net. (See picture below, taken from [A, Fig. 4-9, p.140].) The boundary curve of a square Q in the net is denoted ∂Q. The orientation of the boundary ∂Q is chosen in the usual way, namely that the interior of Q falls to the



[1]This means no two squares have a common interior, and if two squares intersect at more than a vertex then they share an edge.

left of the directed edges of Q. Let Q_a be the square containing a at its centre. And let S be the set of squares $S = \{Q \mid Q \cap A \neq \emptyset\}$. Since A is compact S is finite. Consider the cycle

$$\Gamma_1 = \sum_{Q \in S} \partial Q.$$

We note that

$$\eta(\partial Q, a) = \begin{cases} 1 & \text{if } Q = Q_a \\ 0 & \text{if } Q \in S \text{ and } Q \neq Q_a \end{cases}$$

It follows that

$$(**) \qquad \eta(\Gamma_1,\, a) = 1.$$

Since the distance between A and B is δ and each of the squares in S has side $< \delta/\sqrt{2}$, it is clear that if $Q \in S$ then $Q \cap B = \emptyset$. Thus $\Gamma_1^* \cap B = \emptyset$. Let $Q \in S$. We regard ∂Q as the sum of four oriented edges, namely the edges of Q, with the understanding that the orientation of an edge σ in ∂Q is the one such that Q falls to its left when we traverse it. Let $\Phi_1 = \{\sigma \mid \sigma \text{ is an oriented edge of some } Q \in S\}$. Note that

$$\Gamma_1 = \sum_{E \in \Phi_1} E.$$

Now if E is an oriented edge of a member of S, and $E^* \cap A \neq \emptyset$, then E^* must be the common unoriented edge of two adjacent squares Q and R, and hence the orientation on E^* from Q is opposite to the orientation on E^* from R. The two orientations work to 'cancel' the edge E from the sum representing Γ_1. More precisely if Ψ is the set of oriented edges E in Φ_1 such that $E^* \cap A \neq \emptyset$ then $\sum_{E \in \Psi} E \in \mathfrak{R}(\mathbf{C})$. Thus if $\Phi = \Phi_1 \smallsetminus \Psi$, then the cycle

$$\Gamma = \sum_{E \in \Phi} E$$

is such that $\Gamma - \Gamma_1 \in \mathfrak{R}(\mathbf{C})$. Since $E^* \cap B = \emptyset$ and $E^* \cap A = \emptyset$ for $E \in \Phi$, it follows that Γ is a cycle in Ω. Moroever, since $\Gamma - \Gamma_1 \in \mathfrak{R}(\mathbf{C})$, it follows that $\eta(\Gamma,\, a) = \eta(\Gamma_1,\, a)$. Thus from $(**)$ we get $\eta(\Gamma,\, a) = 1$. We have therefore shown that if Ω is not simply connected then there is a point $a \in \mathbf{C} \smallsetminus \Omega$ and a cycle Γ in Ω such that $\eta(\Gamma,\, a) \neq 0$. $\square$

Remarks 2.1.3. 1) In fact, the latter part of the above argument also shows that if Ω is *classically simply connected* then it is simply connected in our sense, i.e., in the sense of Definition 2.1.1. In fact the universal-coefficients theorem tells us that

$$\mathrm{H}^1(M,\, \mathbf{R}) \xrightarrow{\ \sim\ } \mathrm{Hom}(\pi_1(M,\, p),\, \mathbf{R})$$

for any connected manifold M with base point p. This means that if M has a trivial fundamental group then $\mathrm{H}^1(M,\, \mathbf{R}) = 0$. In particular, if a region Ω in $\mathbf{C}$ is classically simply connected, then $\mathrm{H}^1(\Omega,\, \mathbf{R}) = 0$. Then every closed 1-form ω (always taken to be C^1) is exact, whence $\int_\Gamma \omega = 0$ for every cycle Γ in Ω. In particular, taking $\omega = \frac{1}{2\pi i}(z-a)^{-1}dz$ this gives $\eta(\Gamma,\, a) = 0$ for every $a \notin \Omega$ and every cycle Γ in Ω. By Theorem 2.1.2, we see that Ω is simply connected in our sense also.

2) What we saw in the above remark is that if Ω is a region in $\mathbf{C}$ and $\mathrm{H}^1(\Omega,\, \mathbf{R}) = 0$ then Ω is simply connected. In fact for regions in $\mathbf{C}$, simply connected regions Ω are *exactly* the regions for which $\mathrm{H}^1(\Omega,\, R) = 0$ or for that matter regions such

that $\mathrm{H}^1(\Omega, \mathbf{C}) = 0$. This involves a topic not often taught in Algebraic Topology courses. A consequence of *Alexander Duality* is that for an open set U of $\mathbf{R}^n$ wiith $Z = \mathbf{R}^n \setminus U$, and for any field k, the dimension of the cohomology group $\mathrm{H}^{n-1}(U, k)$ as a k-vector space is the number of *compact* connected components of Z [I, 6.8, p.283]. Specialise to $n = 2$. Now it is obvious that Ω is simply connected in $\mathbf{C}$ (in the sense of Definition 2.1.1) if and only if $\mathbf{C} \setminus \Omega$ has no compact connected components. Thus by the stated result from [I] we have Ω is simply connected if and only if $\mathrm{H}^1(\Omega, \mathbf{R}) = 0$. Exactly the same proof shows that a region Ω in $\mathbf{C}$ is simply connected if and only if $\mathrm{H}^1(\Omega, \mathbf{C}) = 0$.

3) For regions in $\mathbf{C}$ we have just seen that:

(†) *Classically simply connected* $\Rightarrow$ *Simply connected.*

The reverse implication is also true via the *Riemann Mapping Theorem* which we will prove later in the course.

3. The Generalised Cauchy Theorem

The following is a generalisation of Cauchy's Theorem.

Theorem 3.1 (Generalised Cauchy's Theorem). *Let Ω be a region and Γ a cycle in Ω such that $\eta(\Gamma, a) = 0$ for every $a \notin \Omega$. Then*

$$\int_\Gamma f(z)dz = 0$$

for every holomorphic function $f(z)$ on Ω.

References

[A] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New-York, 1979.

[I] B. Iversen *Cohomology of Sheaves*, Universitext, Springer-Verlag, Berlin, 1986.