LECTURE 10

Date of Lecture: February 8, 2017

Some of the proofs here are elaborations and cleaner expositions of what was given in class. Some results were unfortunately not stated in class (see Proposition **??** below). Others are a quick summary.

1. Averaging Property of Harmonic Functions

We stated in Lecture 9 (formula (A_2)) that a harmonic function on an open set in $\mathbf{R}^n$ has the "averaging property". Here is the proof for the case $n = 2$

Theorem 1.1. *Suppose u is harmonic in a region Ω in $\mathbf{C}$. Let $D = B(a, r)$ be a disc of positive radius such that its closure $\overline{D}$ in $\mathbf{C}$ lies in Ω. Then*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta.$$

Proof. From Theorem 2.2.1 of Lecture 9 we know that u has a harmonic conjugate v on $\overline{D}$. Let $f = u + iv$. Then f is analytic on $\overline{D}$. By Cauchy's Theorem

$$\begin{aligned}
u(a) + iv(a) &= \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a} \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}(ire^{i\theta})d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})d\theta + i\frac{1}{2\pi}\int_0^{2\pi} v(a+re^{i\theta})d\theta.
\end{aligned}$$

Equating real and imaginary parts we are done. ☐

2. The fundamental theorem of algebra

We forgot to prove an important consequence of Louisville's theorem

Theorem 2.1. *Let $p(z)$ be polynomial with complex coefficients of degree ≥ 1. Then $p(z)$ has a root in $\mathbf{C}$.*

Proof. Let $\deg p(z) = n$, say

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n$$

with $n \geq 1$, and $a_n \neq 0$. Then on $\mathbf{C}^* := \mathbf{C} \smallsetminus \{0\}$ we have

$$p(z) = z^n g(z)$$

where $g \colon \mathbf{C}^* \to \mathbf{C}$ is the holomorphic function

$$g(z) := \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z} + a_n\right).$$

Since $\lim_{z\to\infty} g(z) = a_n \neq 0$, there exists $R > 0$ such that $|g(z) - a_n| < |a_n|/2$ whenever $|z| \geq R$. This implies that

$$\left|g(z)\right| > \frac{|a_n|}{2} \qquad (|z| \geq R).$$

This means

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n|}\frac{1}{R^n} \qquad (|z| \geq R).$$

It follows that $\lim_{z\to\infty} p(z)^{-1} = 0$. If $p(z)$ has no roots in $\mathbf{C}$, then $h(z) = p(z)^{-1}$ is also entire, and since it has a finite limit at infinity, it is bounded. By Louisville's theorem $h(z)$ is a constant. This means $h(z)$ is identically zero since $\lim_{z\to\infty} h(z) = 0$. But $p(0) = a_n \neq 0$, whence $h(0) = 1/a_n \neq 0$. This is a contradiction. □

3. Simply connected regions

3.1. The definition used by function theorists is the one in Definition 3.1.1 below. We denote the Riemann sphere by $\widehat{\mathbf{C}}$.

Definition 3.1.1. A region Ω in $\mathbf{C}$ is *simply connected* if $\widehat{\mathbf{C}} \smallsetminus \Omega$ is connected.

Remark 3.1.2. The function theory definition is only used for regions in the plane $\mathbf{C}$ and not for arbitrary topological spaces. The definition of simple-connectedness in topology is that Ω is simply connected if every closed path is path homotopic in Ω to the trivial (constant) path. We will call regions *classically simply connected* or *simply connected in the classical sense*. It turns out that the two definitions, *for regions in* $\mathbf{C}$, are equivalent. One way (classical simple-connectedness $\Rightarrow$ simple-connectedness) is easy, using De Rham's theorem and the universal coefficient theorem. In fact using this machinery, one sees that if a region Ω is classically simply connected, then its first De Rham cohomology group $\mathrm{H}^1(\Omega, \mathbf{R}) = 0$. This means every closed 1-form[1] on Ω is exact and hence if ω is a closed 1- form then its line integral along any closed path is zero. Recall that by Cauchy-Riemann equations, forms of the kind $f(z)dz$ with f holomorphic on Ω are closed. In particular if $a \notin \Omega$, we have $\eta(\gamma, a) = 0$ for every closed path in Ω, for $dz/(z-a)$ is a holomorphic form on Ω. We now appeal to Theorem 3.1.3 below to conclude that Ω is simply connected.

The converse needs the Riemann Mapping Theorem which will be proved later in the course.

Theorem 3.1.3. *A region Ω is simply connected if and only if $\eta(\gamma, a) = 0$ for all closed paths γ in Ω and all points $a \in \mathbf{C} \smallsetminus \Omega$.*

Proof. Next lecture. □

References

[A] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New-York, 1979.

[1] We only consider C^1-forms in this remark