LECTURE 1

Date of Lecture: January 5, 2017

Here is a summary of what we did in the first class.

1. Analytic functions

1.1. **Definitions.** Let $A \neq \emptyset$ be a subset of $\mathbf{C}$, $f\colon A \to \mathbf{C}$ a function, and $a \in A$ a point. We say f *has a derivative at* a or f is differentiable at a if a is an interior point of A and the following limit exists

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

If f has a derivative at a, we call the displayed limit the *derivative of* f *at* a and denote it using notations borrowed from any of the standard notations in real analysis, e,g., $f'(a)$, $\frac{df}{dz}(a)$, $\frac{df}{dz}|_{z=a}$, etc.

A function is *analytic*, or *holomorphic*, on an open subset Ω of $\mathbf{C}$ if it is differentiable at every point of Ω.

A function f on an arbitrary non-empty subset A of $\mathbf{C}$ is said to be *analytic on* A if it is defined and analytic on an open neighbourhood of A.

A *region* in $\mathbf{C}$ is a non-empty connected subset of $\mathbf{C}$.

1.2. **Example.** Suppos f is analaytic on a region Ω and f is *real-valued*. We showed that f must be a constant, since the derivative at any point is both real and purely imaginary, and hence zero. This means that the gradient of the real as well as the imaginary part of f are, whence each of these components is constant, forcing f to be a constant.

2. Power Series

2.1. By a power series we mean a formal expression of the form $S(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbf{C}$. It is said to *represent* a holomorphic function $f(z)$ on a region D if for every $w \in D$, $\sum_{n=0}^{\infty} a_n w^n$ converges absolutely and the sum equals $f(w)$. The *radius of convergence* $R = R(S)$ of $S(z)$ is defined to be

$$R = \left(\overline{\lim_n} |a_n|^{\frac{1}{n}}\right)^{-1}.$$

Note that $R \in [0, \infty]$. We say that $S(z)$ is *convergent* if $R(S) \neq 0$, and the (possibly infinite) disc $B(a, R(S)) := \{z \mid |z-a| < R(S)\}$ is called the *disc of convergence* of $S(z)$. The *derived series* of $S(z)$ is defined to be $S'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. It is easy to see, using $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, that $R(S) = R(S')$. We proved the following in class

Theorem 2.1.1. *Let $S(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ be a power series centered at $a \in \mathbf{C}$. Let $R = R(S)$. Then the series $S(z)$ converges absolutely if $|z-a| < R$, and diverges if $|z-a| > R$. In the disc of convergence $B(a, R)$, the convergence is uniform on compact sets. If $f\colon B(a, R) \to \mathbf{C}$ is the function $z \mapsto \sum_{n=0}^{\infty} a_n (z-a)^n$, then f is analytic and f' is represented by the power series $\sum_{n=1}^{\infty} n a_n z^{n-1}$.*

Corollary 2.1.2. *The following formula holds*

$$a_n = \frac{f^{(n)}(a)}{n!} \qquad (n \geq 0).$$

In particular there is at most one power series which represents an analytic function on an open disc.

Remark 2.1.3. In view of the Corollary, there is no longer any need to make a distinction between a convergent power series and the analytic function it represents. If $R(S) > 0$ we will regard $S(z)$ as an analytic function as well as a power series, since there is no ambiguity in so doing.