HW 1

(1) Prove rigorously that the functions $f(z)$ and $\bar{f}(\bar{z})$ are simultaneously analytic. How are the derivatives related when they are analytic?

Solution. The underlying assumption is that f is defined on an open set Ω in the complex plane $\mathbf{C}$. Let $\overline{\Omega} = \{z \in \mathbf{C} \mid \bar{z} \in \Omega\}$ and $g\colon \overline{\Omega} \to \mathbf{C}$ the map $z \mapsto \bar{f}(\bar{z})$. We have to show that f is analytic if and only if g is analytic. Note that if we apply the process we just outlined for f to the function g, we recover the function f. Let $\sigma\colon \mathbf{C} \to \mathbf{C}$ be the conjugation map. Recall σ is continuous, respects addition, respects limits (i.e., commutes with limits), σ^2 is the identity, and $\sigma(0) = 0$. Suppose f is analytic. For $z \in \overline{\Omega}$ we have

$$\begin{aligned}\frac{g(z+h)-g(z)}{h} &= \frac{\bar{f}(\overline{z+h}) - \bar{f}(\bar{z})}{h} \\ &= \sigma\left[\frac{f(\sigma(z)+\sigma(h)) - f(\sigma(z))}{\sigma(h)}\right]\end{aligned}$$

Taking the limit as $h \to 0$, and noting that σ commutes with limits we get

$$\begin{aligned}\lim_{h\to 0}\frac{g(z+h)-g(z)}{h} &= \sigma\left[\lim_{h\to 0}\frac{f(\sigma(z)+\sigma(h)) - f(\sigma(z))}{\sigma(h)}\right] \\ &= \sigma\left[\lim_{\sigma(h)\to 0}\frac{f(\sigma(z)+\sigma(h)) - f(\sigma(z))}{\sigma(h)}\right] \\ &= \sigma(f'(\bar{z})).\end{aligned}$$

Thus the derivative of g exists and is $\overline{f'(\bar{z})}$. Since the roles of f and g are symmetric, by symmetry one sees that if g is analytic on $\overline{\Omega}$ then f is analytic on Ω. □

(2) Find the radius of convergence of the following power series

$$\sum n^p z^n, \sum \frac{z^n}{n!}, \sum n! z^n, \sum z^{n!}$$

Solution. Note that $n^{1/n} \to 1$ as $n \to \infty$. To see this apply L'Hôpital's rule to get $\lim_{n\to\infty}(\log x)/x = \lim_{x\to\infty} \frac{1/x}{1} = 0$, and since e^x is continuous, it follows that $\lim_{x\to\infty} x^{1/x} = 1$.

For the first series, note that

$$\begin{aligned}\overline{\lim_{n\to\infty}}|n^p|^{\frac{1}{n}} &= \overline{\lim_{n\to\infty}}|n^{\frac{1}{n}}|^p \\ &= \left|\lim_{n\to\infty} n^{\frac{1}{n}}\right|^p \\ &= 1.\end{aligned}$$

This gives the radius of convergence of the first series as 1.

For the second, we use the fact that if $\lim|\frac{a_n}{a_{n+1}}| = R$ for a then the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R. Using this one sees that the radius of convergence in this case is ∞.

The third series, by the same trick, yields a radius of convergence equal to 0.

For the last series, let a_n be the coefficient of z^n in the power series $\sum z^{n!}$. Then

$$a_k = \begin{cases} 0 & \text{if } k \neq n! \text{ for any } n \\ 1 & \text{if } k = n! \text{ for some (necessarily unique) } n. \end{cases}$$

It follows that $\overline{\lim}_{n\to\infty}|a_n|^{1/n} = \lim n \to \infty 1^{1/n!} = 1$. The radius of convergence is thus 1. □

(3) If $\sum a_n z^n$ has radius of convergence R, what is the radius of convergence of $\sum a_n z^{2n}$? of $\sum a_n^2 z^n$?

Solution. Let R be the radius of convergence of $\sum_n a_n z^n$.

For the first series, let b_k be the coefficient of z^k. Then

$$b_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ a_{k/2} & \text{if } k \text{ is even.} \end{cases}$$

Thus

$$\begin{aligned}\overline{\lim_{k\to\infty}}|b_k|^{1/k} &= \overline{\lim_{k\to\infty}}|a_{k/2}|^{1/k} \quad (k \text{ even}) \\ &= \overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{2n}} \quad (\text{setting } k = 2n) \\ &= \overline{\lim}\left(|a_n|^{1/n}\right)^{1/2} \\ &= \left(\frac{1}{R}\right)^{1/2}.\end{aligned}$$

Similar reasoning shows that the radius of convergence of $\sum_n a_n^2 z^n$ is R^2. □

(4) Let $\sum a_n z^n$ be a power series and K and k two positive numbers.

(a) Assume that for some complex number z_0, $|a_n z_0|^n < Kn^k$ for all $n \geq 0$. Show that the power series converges for every z such that $|z| < |z_0|$

(b) Do part (a) under the assumption that $|a_0 + a_1 z_0 + \ldots + a_n z_0^n| < Kn^k$ for all $n \geq 0$.

Solution. For part (a) suppose $|z|<|z_0|$. Let $\theta = |\frac{z}{z_0}| < 1$. Then $|a_n z^n| = |a_n z_0^n|\theta^n < Kn^k\theta^n$. The series $\sum n^k\theta^n$ converges for $\theta < 1$. This proves part (a)

For (b) note that $|a_n z_0^n| = |(a_0 + \cdots + a_n z_0^n) - (a_0 + \cdots + a_{n-1} z_0^{n-1}| \leq 2Kn^k$. This reduces to part (a) □

(5) Let γ be a closed path in $\mathbf{C}$ and φ a continuous (complex-valued) function on (the image of) γ. Show that

$$f_n(z) := \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

is holomorphic on the components of $\mathbf{C}$ determined by γ and that

$$f_n'(z) = (n+1) \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+2}} d\zeta.$$

Solution. Let the domain of the closed path γ be the closed interval $[a, b]$. We have a complex measure μ on $[a, b]$ with its standard sigma-algebra (i.e., the Lebesgue sigma-algebra) given by $d\mu(t) = \varphi(\gamma(t))\gamma'(t)dt$. More precisely μ is the measure $E \mapsto \int_E (\varphi \circ \gamma)\gamma' dm$, where m is the Lebesgue measure on $[a, b]$, and E varies over the Lebesgue sigma-algebra. The results then follow trivially from Theorem 1.1 of Lecture 2, with $X = [0, 1]$, $\mathscr{F}$ the Lebesgue sigma-algebra, and μ as we just defined it. □

(6) Compute

$$\int_\gamma x dz$$

where γ is the directed line segment from 0 to $1 + i$.

Solution. The obvious parameterisation for γ is $t \mapsto t + it$ on $[0, 1]$. It follows that

$$\begin{aligned} \int_\gamma x dz &= \int_0^1 t(1+i)dt \\ &= (1+i)\left[\frac{t^2}{2}\right]_{t=0}^1 \\ &= \frac{1}{2}(1+i). \end{aligned}$$

□

(7) Compute

$$\int_{|z|=2} \frac{dz}{z^2 - 1}$$

for the positive sense of the circle.

Solution. We have

$$\frac{1}{z^2 - 1} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right].$$

Now, using the parameterisation $t \mapsto 2e^{it}$, $0 \leq t \leq 2\pi$, we see that

$$\begin{aligned}\int_{|z|=2} \frac{dz}{z-1} &= \int_0^{2\pi} \frac{2ie^{it}dt}{2e^{it}-1} \\ &= i\int_0^{2\pi} \frac{2e^{it}-1}{2e^{it}-1} + i\int_0^{2\pi} \frac{dt}{2e^{it}-1} \\ &= 2\pi i + i\int_0^{2\pi} \frac{dt}{2e^{it}-1}.\end{aligned}$$

Similarly

$$\int_{|z|=2} \frac{dz}{z+1} = 2\pi i - i\int_0^{2\pi} \frac{dt}{2e^{it}+1}.$$

So

$$\int_{|z|=2} \frac{dz}{z^2-1} = i\left\{\int_0^{2\pi} \frac{dt}{2e^{it}-1} + \frac{dt}{2e^{it}+1}\right\}.$$

Now

$$\frac{1}{2e^{it}-1} = \frac{2\cos t - 1}{5-4\cos t} - \frac{2i\sin t}{5-4\cos t}$$

One checks that the integral of $\sin t/(5-4\cos t)$ over $[0,\, 2\pi]$ with respect to dt is zero, by noting that an antiderivative of the integrand is $\frac{1}{4}\ln|5-4\cos t|$. Thus

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{2e^{it}-1} &= \int_0^{2\pi} \frac{2\cos t-1}{5-4\cos t}dt \\ &= \left(\int_0^{\pi} + \int_{\pi}^{2\pi}\right)\left[\frac{2\cos t-1}{5-4\cos t}\right]dt\end{aligned}$$

An easy symmetry argument shows that the two integrals in the last line are equal, whence

$$\int_0^{2\pi} \frac{dt}{2e^{it}-1} = 2\int_0^{\pi} \frac{2\cos t-1}{5-4\cos t}dt.$$

Similarly

$$\int_0^{2\pi} \frac{dt}{2e^{it}+1} = 2\int_0^{\pi} \frac{2\cos t+1}{5+4\cos t}dt.$$

Adding, we get

$$\int_0^{2\pi} \frac{dt}{2e^{it}-1} + \frac{dt}{2e^{it}+1} = 2\int_0^{\pi} \frac{12\cos t}{9+16\sin^2 t}dt.$$

Using the decomposition $\int_0^{\pi} = \int_0^{\pi/2} + \int_{\pi/2}^{\pi}$ and applying the identity $\cos t = -\cos(\pi - t)$ we get (via the substitution $u = \pi - t$)

$$\int_0^{\pi} \frac{12\cos t}{9+16\sin^2 t}dt = 0.$$

It follows that

$$\int_{|z|=2} \frac{dz}{z^2-1} = 0.$$

□

(8) Suppose that $f(z)$ is analytic on a closed curve γ (i.e., f is analytic in a region that contains γ). Show that

$$\int_\gamma \overline{f(z)} f'(z) dz$$

is purely imaginary. (The continuity of $f'(z)$ is taken for granted.)

Solution. Let $f = u + iv$. Then $f\bar{f} = |f|^2 = u^2 + v^2$. Let x and y be the co-ordinates along the real and imaginary axes respectively. Then using the Cauchy-Riemann equations for the third equality below, we get

$$\begin{aligned} \overline{f(z)} f'(z) dz &= (u - iv)(u_x + iv_x)(dx + idy) \\ &= [(uu_x + vv_x) + i(uv_x - vu_x)](dx + idy) \\ &= [(uu_x + vv_x) - i(uu_y + vv_y)](dx + idy) \\ &= (uu_x + vv_x)dx + (uu_y + vv_y)dy \\ &\quad + i[(uu_x + vv_x)dy - (uu_y + vv_y)dx] \\ &= \frac{1}{2} d(u^2 + v^2) + i\omega \end{aligned}$$

where $\omega = (uu_x + vv_x)dy - (uu_y + vv_y)dx$. Since $d(u^2 + v^2)$ is an exact differential and γ is a closed path, we have $\int_\gamma d(u^2 + v^2) = 0$. Thus

$$\int_\gamma \overline{f(z)} f'(z) dz = i \int_\gamma \omega.$$

Since ω is a real form, the number on the right side is purely imaginary. $\square$