HW 9

Due on April 3, 2017 (in class).

We use the following convention. If $u\colon \Delta \to \mathbf{C}$ is a map then for $0 \le r < 1$, then $u_r\colon \mathbf{T} \to \mathbf{C}$ is the function given by the formula

$$u_r(e^{it}) = u(re^{it}) \qquad (0 \le r < 1,\, t \in \mathbf{R}).$$

In the problems that follow, you may use the following facts from Fourier theory.

I. The formal Fourier series $\sum_{n\in\mathbb{Z}} a_n e^{int}$ defines an element of $\varphi \in L^2(\mathbf{T})$ if and only if $\{a_n\} \in L^2(\mathbb{Z})$, where $L^2(\mathbb{Z})$ is the Hilbert space of square integrable functions on $(\mathbb{Z}, \#)$ with $\#$ is the counting measure. In this case $\widehat{\varphi}(n) = a_n$ for $n \in \mathbb{Z}$ and we have the Parseval formula

$$\sum_{n\in\mathbb{Z}} |a_n|^2 = \|\varphi\|_{L^2(\mathbf{T})}.$$

II. If $\{P_r\}_{0\le r<1}$ is the Poisson kernel, then

$$\widehat{P}_r(n) = r^{|n|} \qquad (n \in \mathbb{Z}).$$

III. If $\varphi \in L^2(\mathbf{T})$ and $u = P[\varphi]$, then

$$\widehat{u}_r(n) = r^{|n|}\widehat{\varphi}(n) \qquad (n \in \mathbb{Z}).$$

Problems.

(1) Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on Δ. Show that

$$\widehat{f}_r(n) = \begin{cases} 0 & \text{when } n < 0, \\ a_n r^n & \text{when } n \ge 0. \end{cases}$$

(2) Suppose $u\colon \Delta \to \mathbf{C}$ is harmonic (i.e., its real and imaginary parts are harmonic). Show that

$$a_n := \frac{\widehat{u}_r(n)}{r^{|n|}} \qquad (n \in \mathbb{Z})$$

is independent of r, $0 \le r < 1$. [Hint: Fix $s \in [0,1)$. Let $g = P[u_s]$. Show that for $0 \le r \le s$, $g_{\frac{r}{s}} = u_r$.]

(3) (Fatou's Theorem for bounded harmonic functions) Let u be as above, and suppose further that u is *bounded*.

(a) If a_n, $n \in \mathbb{Z}$, are as above, show that $\{a_n\} \in L^2(\mathbb{Z})$.

(b) Show that $u = P[\varphi]$ for a unique φ in $L^2(\mathbf{T})$.

(c) Show that for almost all $e^{i\theta} \in \mathbf{T}$, $\lim_{r\to 1^-} u(re^{i\theta})$ exists, i.e., *radial limits exist for u for almost all θ.* (In particular, if f is a bounded holomorphic function on Δ, then f has almost all radial limits. This is Fatou's theorem for holomorphic functions.)