HW 7

Due on March 8, 2017 (in class).

Rational functions.

(1) Suppose that $R(z) = P(z)/Q(z)$ is a rational function, where we assume that $P(z)$ and $Q(z)$ are polynomials with no common factors. Let λ be equal to the larger of the degrees of P and Q. We view $R(z)$ as a mapping of the extended complex plane to itself (i.e., from the Riemann sphere to itself). In this setting, show that the total number of zeros of R is equal to λ. Show that the number of poles is also λ. Show that, for all but a finitely many values of a, there are exactly λ *distinct* solutions to $R(z) = a$.

(2) Suppose P and Q are polynomials of the same degree with no common factors. Let K denote the compact set consisting of the closed line segments which join the zeros of P and Q to the origin. Prove that there exists a branch of $\log \frac{P(z)}{Q(z)}$ defined on $\mathbf{C} \setminus K$.

Rouché's Theorem.

(3) Use Rouché's Theorem to prove that if f is analytic on $\{z \mid |z| \leq 1\}$, and if $|f(e^{i\theta})| < 1$ for $0 \leq \theta \leq 2\pi$, then f has exactly one fixed point in the disc $B(0, 1)$.

(4) Use Rouché's Theorem to prove that there are no polynomials of the form

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

satisfying $|p(z)| < 1$ for all z with $|z| = 1$.

Implicit function theorem. For a function of two variables (z, w), we prefer to use the symbols ∂_1 and ∂_2 for the partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$.

Let $p(z, w)$ be analytic in both variables (i.e., for fixed w it is analytic in z and vice-versa) on $U \times V \subset \mathbf{C}^2$, where U and V are regions in $\mathbf{C}$. Let $(z_0, w_0) \in U \times V$ be a point such that $p(z_0, w_0) = 0$ and $\partial_2(p(z_0, w_0)) \neq 0$. For fixed $z \in U$, let $f_z \colon V \to \mathbf{C}$ be the holomorphic function $f_z(w) = p(z, w)$.

In the following you may find problem 2 of HW 5 and problem 6 of HW 6 useful.

(5) Show that there is a circle C centred at w_0, with the closed disc enclosed by C contained in V, and an open neighbourhood Ω of z_0, $\Omega \subset U$, such that for each fixed $z \in \Omega$, $\eta(f_z(C), z) = 1$.

(6) Let D be the open disc enclosed by the circle C of the previous problem. Let Ω also be as in the previous problem. Show that there is a holomorphic function $g\colon \Omega \to D$ such that $g(z_0) = w_0$, and $p(z,\, g(z)) = 0$, $z \in D$. Moreover show that for each $z \in D$, $w = g(z)$ is the only solution of $p(z, w) = 0$ in D.

Simply connected regions. As usual, we follow the function theoretic definition of simple connectedness.

(7) Show *using only Theorem 2.1.2 of Lecture 11, and Problem 10 of HW 6*, that a classically simply connected region is simply connected. [Hint: The last sentence of Problem10 (c) of HW 6 may be useful.]

(8) Show that a harmonic function on a simply connected region has a conjugate.

Harmonic functions.

(9) Suppose u is continuous on $\{z \mid |z| \leq 1\}$ and that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + (1 - |z|)e^{i\theta})d\theta \quad \text{for all } z \in B(0, 1).$$

Prove that u is harmonic in $B(1, 0)$.

(10) Let u be harmonic on a region Ω. Show that the points where the gradient of u is zero must be isolated unless u is constant.