HW 6

Due on March 1, 2017 (in class).

Harmonic Functions.

(1) Suppose u is harmonic in a region Ω which contains 0, and the disc $|z| \leq R$ is contained in Ω.

(a) Show that (with $z = Re^{i\theta}$)

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z) d\theta$$

for all $|a| < R$. [Hint: For a such that $|a| < R$, consider the map T given by $z \mapsto R(Rz + a)/(R + \bar{a}z)$. Show that T maps the unit disc bijectively on to the disc $\{|z| \leq R\}$ and sends 0 to a. Apply the averaging property for a suitable transformation of u.]

(b) We assumed in part (a) that u was harmonic on the closed disc $\overline{D} = |z| \leq R$. Show that the assumption can be relaxed in the following way. Suppose u is continuous on $\overline{D}$ and harmonic in the interior D of $\overline{D}$. Show that the formula in part (a) continues to hold. [Hint: For $0 < r < 1$, consider the function $z \mapsto u(rz)$. Take appropriate limits. Show that the limit passes through an integral sign.]

(2) Show that the uniform limit of harmonic functions is harmonic.

(3) Let u be harmonic in a region Ω.

(a) If u attains a maximum or a minimum in Ω, show that u must be a constant.

(b) Show that if u_1 and u_2 are harmonic in a region containing a closed disc $\overline{D}$ of positive radius, and if u_1 and u_2 agree on the bounding circle of $\overline{D}$, then $u_1 = u_2$ on $\overline{D}$.

Generalised Cauchy's formula. In complex analysis, the following definition is used for cycles homologous to zero in a region. *Let Ω be a region. A cycle Γ in Ω is said to be homologous to zero in Ω if $\eta(\Gamma, a) = 0$ for every $a \notin \Omega$.*

If Γ is homologous to zero in Ω, we often write $\Gamma \sim 0 \pmod{\Omega}$. If Γ and Γ' two cycles in Ω and $\Gamma - \Gamma' \sim 0 \pmod{\Omega}$, we often write $\Gamma \sim \Gamma' \pmod{\Omega}$ and say Γ is homologous to Γ' with respect to Ω. Note that if $\Gamma \sim 0 \pmod{\Omega}$ then $\Gamma \sim 0$ (mod Ω') for every $\Omega' \supset \Omega$.

Assume the Generalised Cauchy-Goursat Theorem, namely that $\int_\Gamma f(z)dz = 0$ for holomorphic functions $f(z)$ on a region Ω and cycles $\Gamma \sim 0 \pmod{\Omega}$.

(4) Let Ω be a simply connected[1] region. Let $f(z)$ be analytic in Ω.

(a) Show that $\int_\Gamma f(z)dz = 0$ for every cycle Γ in Ω.

(b) If $f(z)$ is nowhere vanishing on Ω, show using the Generalised Cauchy-Goursat theorem that it is possible to define a single-valued analytic branch of $\log f(z)$ in Ω, i.e., it is possible to find an analytic function $g(z)$ on Ω such that $e^{g(z)} = f(z)$.

(c) Let $f(z)$ be as in part (b). Show that it is possible to define a single-valued branch of $\sqrt[n]{f(z)}$.

(d) Suppose f has a zero of multiplicity m at $z_\circ$. Show that there there is a small disc $D = B(z_\circ, \epsilon)$ on which an analytic function $g(z)$ can be defined such that $g(z)^m = f(z)$.

The Argument Principle.

(5) If $f(z)$ is meromorphic in a region Ω with zeros a_i and poles b_k, show that

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = \sum_i \eta(\Gamma, a_i) - \sum_k \eta(\Gamma, b_k)$$

for every cycle $\Gamma \sim 0 \pmod{\Omega}$ with the property that Γ^* does not contain any of the zeros or poles. (The multiple zeros and poles have to be repeated as many times as their order indicates; and the sum is finite.)

(6) Let $D = B(a, r)$ where $r > 0$ and let $\overline{D}$ be its closure. Suppose $f(z)$ is a holomorphic function on $\overline{D}$ such that $f(a) = 0$ and a is the only solution of $f(z) = 0$ in $\overline{D}$. Suppose further that $f'(a) \neq 0$. Let $C = \partial\overline{D}$ with the usual orientation.

(a) Show that

$$\frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz = a.$$

(b) Show, without using the inverse function theorem, that there is a non-empty open subset W of $f(D)$ containing 0 such that for $w \in W$, if $g(w)$ be given by

$$g(w) = \frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z) - w}dz$$

then $g(f(z)) = z$ for $z \in g(W)$ and $f(g(w)) = w$ for $w \in W$.

(c) Show using the integral formula for $g(w)$ in part (b) that g is holomorphic. [Hint: You have to argue that you can differentiate under the integral sign.]

(7) Let $\pi/4 < r < \pi/2$. Let D be an open disc of radius r centred at 0 and C be the circle of radius r centred at 0.

[1] In the function-theoretic sense, i.e., in the sense of the definition used in this course

(a) Show that $\int_C \cot z dz = 2\pi i$.

(b) Show that

$$\int_C \frac{z\cos z}{\sin z - 0.5} dz = \frac{\pi^2 i}{3}.$$

(8) Suppose $P(z, w)$ is a polynomial of degree n in z and degree m in w, i.e.,

$$P(z,\, w) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} z^i w^j$$

where $a_{nj} \neq 0$ for some j and $a_{im} \neq 0$ for some i. Prove that if the polynomial in z given by $p_\circ(z) = P(z, w_\circ)$ has exactly n distinct zeros, then there is an $\epsilon > 0$ such that the polynomial in z given by $p_w(z) = P(z, w)$ has the same property for each fixed w in $B(w_\circ, \epsilon)$.

Integrals. Here are some exercises on integrals and covering spaces.

(9) For $r > 0$, let C_r will denote the *semi-circle* $z(t) = re^{it}$, $0 \leq t \leq \pi$.

(a) Show that

$$\lim_{r\to\infty} \left| \int_{C_r} \frac{1 - e^{2iz}}{z^2} dz \right| = 0.$$

(b) Show that

$$\lim_{r\to 0} \int_{C_r} \frac{1 - e^{2iz}}{z^2} dz = 2\pi.$$

(10) This is an exercise on covering spaces as well as integrals. Let $\mathbf{C}_a^*$ be the complex plane punctured at $a \in \mathbf{C}$, i.e., $\mathbf{C}_a^* = \mathbf{C} \smallsetminus \{a\}$. If $a = 0$, write $\mathbf{C}^*$ for $\mathbf{C}_0^*$. For a path $\gamma\colon [\alpha,\, \beta] \to \mathbf{C}_a^*$, and for $\alpha \leq t \leq \beta$, let $\gamma_{_t} = \gamma|_{[\alpha,\, t]}$.

(a) Show that $f_a\colon \mathbf{C} \to \mathbf{C}_a^*$, $z \mapsto a + e^z$ is the universal covering space of $\mathbf{C}_a^*$. [Hint: Without loss of generality assume $a = 0$ and show that for any $\theta_\circ \in \mathbf{R}$, f_a maps $\{z \in \mathbf{C} \mid \theta_\circ < \mathrm{Im}(z) < \theta_\circ + 2\pi\}$ bijectively on to the $\mathbf{C}^* \smallsetminus L_{\theta_\circ}$ where $L_{\theta_\circ}$ is the ray which makes an angle of $\theta_\circ$ with the positive real axis.]

(b) Show that if $\gamma\colon [\alpha,\, \beta] \to \mathbf{C}_a^*$ is a path staring at $z_\circ \in \mathbf{C}_a^*$, then for every point $w_\circ$ in the fibre $f_a^{-1}(z_\circ)$, the map $\widetilde{\gamma}\colon [\alpha,\, \beta] \to \mathbf{C}$ given by

$$t \mapsto w_\circ + \int_{\gamma_t} \frac{dz}{z - a} \qquad (t \in [\alpha,\, \beta])$$

is the unique path lift of γ for the covering map f_a starting at $w_\circ$. [Hint: See proof of Thm. 1.1 of Lecture 7.]

(c) Show that (with the notations of the part (b)) if a closed path γ in $\mathbf{C}_a^*$ is such that

$$\eta(\gamma,\, a) = n,$$

and $\widetilde{\gamma}$ is a lift of γ to the universal covering space $f_a\colon \mathbf{C} \to \mathbf{C}_a^*$, then

$$\widetilde{\gamma}(\beta) = \widetilde{\gamma}(\alpha) + 2\pi i n.$$

Conclude that γ is path homotopic to the trivial path if and only if $n = 0$.