HW 5

Due on Feb 9, 2017 (in tutorial).

Harmonic Functions.

(1) Find the most general harmonic function of the form $ax^3+bx^2y+cxy^2+dy^3$ on the plane. Determine the conjugate and the corresponding analytic function.

Derivatives of integrals.

(2) Let Ω be a region in $\mathbf{C}$ and $I = [\alpha,\, \beta]$ a closed interval in $\mathbf{R}$. Let $\varphi\colon \Omega\times I \to \mathbf{C}$ be a continuous function. Suppose further that $\varphi(z,\, t)$ is analytic as a function of $z \in \Omega$ for any fixed $t \in I$. Show that

$$F(z) = \int_{\alpha}^{\beta} \varphi(z,\, t)dt$$

is analytic in z and

$$F'(z) = \int_{\alpha}^{\beta} \frac{\partial\varphi(z,\, t)}{\partial z}dt.$$

[**Hint:** Represent $\varphi(z,\, t)$ as a Cauchy integral, and realise $F(z)$ as an iterated integral. Also use an earlier problem you did, which gives the formula for a derivative as an integral.]

Isolated Singularities and meromorphic functions. Suppose f is analytic on $\{z \mid |z| > R\}$. Then $f(\frac{1}{z})$ has an isolated singularity at $z = 0$. The type of singularity of f at ∞ is defined to be the type of the singularity of $f(\frac{1}{z})$ at $z = 0$.

A function $f\colon \Omega \to \mathbf{C}\cup\{\infty\}$ is said to be meromorphic on a region Ω if, for each point $z_0 \in \Omega$, there is a disc $B(z_0,\, r) \subset \Omega$ and functions $G(z)$ and $H(z)$ which are analytic on $B(z_0,\, r)$ and such that H is not identically 0 and $f(z) = G(z)/H(z)$ on $B(z_0,\, r)$.

(3) This problem involves the behaviour of entire functions at ∞.

(a) Describe the set of entire functions that have a removable singularity at ∞.

(b) Describe the set of entire functions that have a pole at ∞.

(4) This problem concerns meromorphic functions.

(a) Prove that if f is meromorphic on Ω, then there is a sequence of points $\{a_i\}_{i=1}^{\infty}$ in Ω with no accumulation points in Ω such that $a_i \neq a_j$ if $i \neq j$, f is analytic on $\Omega \setminus \{a_i \mid i = 1, 2, \ldots\}$ and f has poles at the a_i's. Prove the converse.

(b) Define what it means for a function to be meromorphic on $\mathbf{C} \cup \{\infty\}$. Show that all such meromorphic functions are rational functions.

(5) Prove that an isolated singularity of $f(z)$ cannot be a pole of $e^{f(z)}$.

General Cauchy's Theorem. Assume the following general form the Cauchy's Theorem (we will prove an even more general form later in the course):

Suppose Ω is a region and γ a closed path in Ω such that $\eta(\gamma, a) = 0$ for every $a \notin \Omega$. Then

$$\int_{\gamma} f(z)dz = 0$$

for every holomorphic function on Ω.

(6) Let Ω be a region and γ a closed path in Ω satisfying the hypotheses of the general form of Cauchy's theorem given above. Suppose Ω' is an region containing γ^* and such that $\Omega' \cap U = \Omega \cap U$ for every connected component U of $\mathbf{C} \setminus \gamma^*$ such that $\eta(\gamma, a) \neq 0$ for $a \in U$. Show that

$$\int_{\gamma} f(z)dz = 0$$

for every holomorphic function f on Ω'.

(7) Let Ω be a region and γ a closed path in Ω satisfying the hypotheses of the general form of Cauchy's theorem given above. Let $f(z)$ be analytic on Ω and let $S = \{a_i\}$ be the collection of zeroes of f in Ω. For each $a_i \in S$ let n_i be the order of the zero of f at a_i. Suppose that $S \cap \gamma^* = \emptyset$. Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{i} \eta(\gamma, a_i) n_i.$$

[*Remark and Hint:* Note that S need not be finite. Check that only a finite number terms in the sum on the right side of the above equality are non-zero. Now imitate the proof of the Argument Principle given in class, and use the results of the previous problem to get the solution.]