HW 4

Not to be handed over.

Throughout $\Delta = B(0,1)$. A *conformal map* on a region Ω (for this course) is an analytic map whose derivative is nowhere vanishing on its domain. A *univalent map* on a region Ω is a one-to-one and analytic map.

(1) Suppose that an analytic function is written in polar form via

$$f(re^{i\theta}) = u(r,\,\theta) + iv(r,\,\theta).$$

Derive the polar form of the Cauchy-Riemann equations and use them to verify that the function $\log(r)+i\theta$ is analytic on $\{re^{i\theta} \mid r > 0, -\pi < \theta < \pi\}$.

(2) Suppose $f(z)$ is an analytic function on Δ such that $f'(0) \neq 0$. Prove that there exists an open set U of Δ, with $0 \in U$, and an open set V of $\mathbf{C}$ with $0 \in V$ such that $f(U) = V$ and f is one-to-one on U. Show that the inverse of $U \xrightarrow{f|_U} V$ is analytic on V.

(3) Prove that if f_n is a sequence of univalent function on a region Ω which converges uniformly on compact subsets to a function f, then either f is constant or univalent on Ω.

(4) A function is said to be an automorphism of a region Ω if it is a one-to-one analytic mapping of Ω onto itself. Prove that if f is an automorphism of a bounded region Ω then (with $b\Omega$ defined to be the boundary of Ω) $f(z) \to b\Omega$ as $z \to b\Omega$.

(5) Suppose f and g are analytic mappings of Δ into a region Ω, f is univalent $f(\Delta) = \Omega$, and $f(0) = g(0)$. Prove that $g(B(0,r)) \subset f(B(0,r))$ for $0 < r < 1$.

(6) Suppose Ω is a region, $z_0 \in \Omega$ and f and g are one-to-one analytic mappings of Ω to Δ such that $f(z_0) = g(z_0) = 0$. What relationship exists between f and g? Answer the same question when $f(z_0) = g(z_0) = a$ for some $a \in \Delta$.

(7) Show that if f is analytic on the unit disc and continuous on the closure of the unit dic with $|f(z)| = c$ for all z with $|z| = 1$, then f is rational.