HW 3

Due on Jan 25, 2017 (in class).

Power Series.

(1) Given a power series $\sum_n a_n z^n$, show that the radius of convergence R is equal to

$$\sup\{r \mid \text{there is a constant } M \text{ such that } |a_n| \leq Mr^{-n},\ \forall n \geq 0\}.$$

Integrals. You will not really need brute force computations for these problems, but some computations may be needed.

(2) Evaluate

(a)

$$\int_{\{|z|=1\}} \frac{e^z}{z} dz.$$

(b)

$$\int_{\{|z|=1\}} e^z z^{-n} dz.$$

(c)

$$\int_{\{|z|=2\}} z^n (1-z)^m dz.$$

Estimates.

(3) Prove that a function that is analytic in the whole plane and satisfies the inequality $|f(z)| < |z|^n$ for some n, and sufficiently large $|z|$, is a polynomial.

(4) If $f(z)$ is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \leq \rho < R$.

Schwarz's Lemma. Throughout $\Delta = \{z \colon |z| < 1\}$.

(5) Suppose $f \colon \Delta \to \Delta$ is analytic and one-to-one. Prove that

$$f(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}$$

for some real θ, and some a, $|a| < 1$.

(6) Prove the following generalised version of Schwarz's lemma: If $f\colon \Delta \to \Delta$ is an analytic map and if $f(a) = 0$ for some $a \in \Delta$, then

$$|f(z)| \leq \frac{|z-a|}{|1-\bar{a}z|} \qquad (z \in \Delta).$$

What can you say if equality occurs for some $z \neq a$?

(7) Show that $|f(z)| \leq 1$ for $|z| \leq 1$ implies that

$$\frac{|f'(z)|}{(1-|f(z)|^2)} \leq \frac{1}{1-|z|^2}.$$