HW 1

(1) Prove rigorously that the functions $f(z)$ and $\bar{f}(\bar{z})$ are simultaneously analytic. How are the derivatives related when they are analytic?

Power Series.

(2) Find the radius of convergence of the following power series

$$\sum n^p z^n, \sum \frac{z^n}{n!}, \sum n! z^n, \sum z^{n!}$$

(3) If $\sum a_n z^n$ has radius of convergence R, what is the radius of convergence of $\sum a_n z^{2n}$? of $\sum a_n^2 z^n$?

(4) Let $\sum a_n z^n$ be a power series and K and k two positive numbers.

(a) Assume that for some complex number z_0, $|a_n z_0|^n < Kn^k$ for all $n \geq 0$. Show that the power series converges for every z such that $|z| < |z_0|$

(b) Do part (a) under the assumption that $|a_0 + a_1 z_0 + \ldots + a_n z_0^n| < Kn^k$ for all $n \geq 0$.

Complex Integration.

(5) Let γ be a closed path in $\mathbf{C}$ and φ a continuous (complex-valued) function on (the image of) γ. Show that

$$f_n(z) := \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

is holomorphic on the components of $\mathbf{C}$ determined by γ and that

$$f_n'(z) = (n+1) \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+2}} d\zeta.$$

(6) Compute

$$\int_\gamma x dz$$

where γ is the directed line segment from 0 to $1 + i$.

(7) Compute

$$\int_{|z|=2} \frac{dz}{z^2 - 1}$$

for the positive sense of the circle.

(8) Suppose that $f(z)$ is analytic on a closed curve γ (i.e., f is analytic in a region that contains γ). Show that

$$\int_\gamma \overline{f(z)} f'(z) dz$$

is purely imaginary. (The continuity of $f'(z)$ is taken for granted.)