Name: ________________________________

GRADUATE ALGEBRA-I
QUIZ 1

Throughout k is a field, V a finite dimensional vector space over k, say $\dim_k V = n$, and $T\colon V \to V$ a linear transformation. Let $k[X]$ be the polynomial ring over k in one variable. For any polynomial $p(X) \in k[X]$, $p(T)\colon V \to V$ has the usual meaning. Define $k[X] \times V \to V$ by the rule $(p(X), v) \mapsto p(T)v$. It is easy to see, and we will assume this in this quiz, that V becomes a $k[X]$-module via this operation.

(1) Without using the Cayley-Hamilton theorem, show that $V_{\text{tor}} = V$.

(2) Suppose $n = 5$, $\lambda, \tau \in k$ are district elements, and the matrix of T with respect to some basis is

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \tau & 1 \\ 0 & 0 & 0 & 0 & \tau \end{bmatrix}$$

(a) Find $\operatorname{Supp}(V) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a non-zero prime ideal of } k[X] \text{ such that } \Gamma_{\mathfrak{p}}(V) \neq 0\}$.

(b) For each $\mathfrak{p}$ in $\mathrm{Supp}(V)$ write $\Gamma_{\mathfrak{p}}(V)$ as a direct sum of the form $\bigoplus_{i=1}^{r} k[X]/\mathfrak{p}^{\mu_i}$. Give a proof that your decomposition is correct. (The μ_i and r depend on $\mathfrak{p}$).

(3) Suppose k is algebraically closed For $\lambda \in k$ and $d \in \mathbb{N}$ the *Jordan matrix* $J_d(\lambda)$ is defined to be the $d \times d$ matrix

$$\begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$$

i..e., $J_g(\lambda)$ has λ on the diagonal spots, 1 along the super-diagonal, and zero elsewhere.

(a) Show that there exists a basis of V with respect to which the matrix of T is

$$\begin{bmatrix} J_{d_1}(\lambda_1) & 0 & 0 & 0 \\ 0 & J_{d_2}(\lambda_2) & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & J_{d_r}(\lambda_r) \end{bmatrix}$$

(b) Show that this representation of T as a matrix is unique up to a permutation of the Jordan blocks $J_{d_1}(\lambda_1), J_{d_2}(\lambda_2), \ldots, J_{d_r}(\lambda_r)$

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