# QUIZ 0

(1) Let $A$ be a ring and $M$ an $A$-module. Define what it means for a subset $N$ of $M$ to be a submodule of $M$. (2 marks)

(2) Let $A$ be an integral domain and $M$ an $A$-module. Show that $M_{\text{tor}}$ is a submodule of $M$. (2 marks)

(3) Let $A$ be a ring. Show:
   (a) $0x = 0$, $\forall x \in A$. (2 marks)
   (b) $(-x)y = -(xy)$ $\forall x, y \in A$. (2 marks)
   (c) Let $M$ be an $A$-module. For $a \in A$ and $x \in M$ show that $a(-x) = -ax$ and $0x = 0$. (2 marks)

**Solutions.** Note, if $A$ is a ring and $M$ an $A$-module, we will sometimes use the notation $0_M$ to denote the additive identity on $M$ and at other times just use 0 to denote this element.

(1) $N$ is a submodule of $M$ if it is an additive subgroup of $M$, and for every $a \in A$ and $x \in N$ we have $ax \in N$.

[*Comments:* There is no need to require distributive properties, since these are inherited from $M$. From the definition given, it is clear that $N$ is an $A$-module in its own right. Some of you wrote that $N$ is a submodule if it is an $A$ module by itself. That is not true unless you require the addition and scalar multiplication to be inherited from $M$. Consider $A = \mathbb{Z}$ and $M = \mathbb{Z}/3\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}\}$. Since $M$ is an abelian group and every abelian group is a $\mathbb{Z}$-module, $M$ is an $A$-module. Consider the subset $N = \{\bar{0}, \bar{1}\}$. Define addition on $N$ by the rules for $\mathbb{Z}/2\mathbb{Z}$. Then $N$ is an abelian group, and hence an $\mathbb{Z}$-module. But it is not a submodule of $M$.]

(2) First let us show that $a0_M = 0_M$ for every $a \in A$. We have

$$\begin{aligned} a0_M &= a(0_M + 0_M) \\ &= a0_M + a0_M. \end{aligned}$$

Now use cancellation in the additive group $M$ to arrive at the required conclusion. Now suppose $x, y \in M_{\text{tor}}$. By definition of $M_{\text{tor}}$, there exist *non-zero* elements $a, b \in A$ such that $ax = 0$ and $by = 0$. Since $A$ is an integral domain therefore $ab \neq 0$. Let $c = ab$. Then

$$\begin{aligned} c(x-y) &= cx - cy \text{ (why?)} \\ &= (ba)x - (ab)y \\ &= b(ax) - a(by) \\ &= b0 - a0 \\ &= 0 \text{ (from what we proved earlier)}. \end{aligned}$$

It follows (since $c \neq 0$) that $x - y \in M_{\text{tor}}$. Thus $M_{\text{tor}}$ is an additive subgroup of $M$. Moreover, if $a \in A$ and $x \in M_{\text{tor}}$, then picking $0 \neq b \in A$ such that $bx = 0$ (such a $b$ exists by definition of $M_{\text{tor}}$) we see that $b(ax) = a(bx) =$

*Date*: August 7, 2015.

$a0 = 0$. Note that the last equality requires us to apply the result we proved earlier, namely $a0_M = 0_M$ for all $a \in A$. Thus $M_{\text{tor}}$ is a submodule of $M$.

(3) (a) $x = 1x = (0 + 1)x = 0x + x$. By cancellation $0x = 0$.
(b) $0 = 0y = (-x + x)y = (-x)y + (xy)$. Thus, $(-x)y$ is an additive inverse for $xy$, i.e., $-(xy) = (-x)y$.
(c) We will use what we proved earlier in our solution to problem 2, namely $a0_M = 0_M$ for every $a \in A$. Thus $0 = a0 = a(-x + x) = a(-x) + ax$. Thus $a(-x)$ is the additive inverse of $ax$. That was the assertion.