MAPPING CONES

1. **Connecting Maps again**

1.1. **Kernels, cokernels and direct sums.** Let $\mathscr{A}$ be ab abelian category.

(1) Let $f\colon A \to W$ be a map in $\mathscr{A}$ and let $(K, i) = \ker(f)$. For any object B in $\mathscr{A}$, we have

$$\ker\left(A \oplus B \xrightarrow{\begin{pmatrix} f & 0 \end{pmatrix}} W\right) = \left(K \oplus B, \begin{pmatrix} i & 0 \\ 0 & 1_B \end{pmatrix}\right) \tag{1.1.1}$$

This is seen as follows. First note that $\begin{pmatrix} i & 0 \\ 0 & 1_B \end{pmatrix}$ is a monomorphism. Indeed if T is an object in $\mathscr{A}$ and $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} a' \\ b' \end{pmatrix}$ are two maps from T to A such that

$$\begin{pmatrix} i & 0 \\ 0 & 1_B \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 1_B \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix}$$

then $i \circ a = i \circ a'$ and $b = b'$. Since i is a monomorphism it follows that $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$.

Next, note that $\begin{pmatrix} f & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & 1_B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Finally, suppose

$$\begin{pmatrix} a \\ b \end{pmatrix} : T \to A \oplus B$$

is a map such that that $\begin{pmatrix} f & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$. Then $f \circ a = 0$ and by the universal property of (K, i) we get a unique map $k\colon T \to K$ such that $i \circ k = a$. It follows that

$$\begin{pmatrix} i & 0 \\ 0 & 1_B \end{pmatrix}\begin{pmatrix} k \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Since $\begin{pmatrix} i & 0 \\ 0 & 1_B \end{pmatrix}$ is a monomorphism, $\begin{pmatrix} k \\ b \end{pmatrix}$ is the only map satisfying the above equation. This proves the assertion.

(2) By duality we see that if $f\colon W \to A$ is a map in $\mathscr{A}$ and $(C, p) = \operatorname{coker}(f)$ then

$$\operatorname{coker}\left(W \xrightarrow{\begin{pmatrix} f \\ 0 \end{pmatrix}} A \oplus B\right) = \left(C \oplus B, \begin{pmatrix} p & 0 \\ 0 & 1_B \end{pmatrix}\right). \tag{1.1.2}$$

1.2. Let $\mathscr{C}$ be an exact category. Suppose we have a short exact sequence of complexes of objects in $\mathscr{C}$:

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

Fix $n \in \mathbb{Z}$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A^n \arrow[r] \arrow[d, "\partial^n"'] & B^n \arrow[r] \arrow[d, "\partial^n"'] & C^n \arrow[r] \arrow[d, "\partial^n"'] & 0 \\
0 \arrow[r] & A^n \arrow[r] & B^n \arrow[r] & C^n \arrow[r] & 0
\end{tikzcd} \tag{1.2.1}$$

With the notations used in notes7.pdf, the connecting map

$$c_n \colon \mathrm{H}^n(C^\bullet) \to \mathrm{H}^{n+1}(A^\bullet)$$

is the map one gets by applying the Snake Lemma to the exact commutative diagram

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
\widetilde{F} \arrow[r] \arrow[d] & \widetilde{Z}^n(A^\bullet) \arrow[r] \arrow[d, "\partial^n_*"'] & \widetilde{Z}^n(B^\bullet) \arrow[r] \arrow[d, "\partial^n_*"'] & \widetilde{Z}^n(C^\bullet) \arrow[r] \arrow[d, "\partial^n_*"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & Z^{n+1}(A^\bullet) \arrow[r] & Z^{n+1}(B^\bullet) \arrow[r] & Z^{n+1}(C^\bullet) \arrow[r] & F \\
0 & & & &
\end{tikzcd} \tag{1.2.2}$$

In particular, if $(K, u) = \ker\left(\widetilde{Z}^n(B^\bullet) \to Z^{n+1}(C^\bullet)\right)$ and $(K', v) = \operatorname{coker}\left(\widetilde{Z}^n(A^\bullet) \to Z^{n+1}(B^\bullet)\right)$ and $\varphi \colon K \to K'$ is the composite $\varphi = v \circ \partial^n_* \circ u$, then c_n is characterised by the commutativity of

$$\begin{tikzcd}
K \arrow[r, "\varphi"] \arrow[d, two heads] & K' \\
\mathrm{H}^n(C^\bullet) \arrow[r, "c_n"'] & \mathrm{H}^{n+1}(A^\bullet) \arrow[u, hook]
\end{tikzcd}$$

On the other hand, from (1.2.1) we have another connecting map

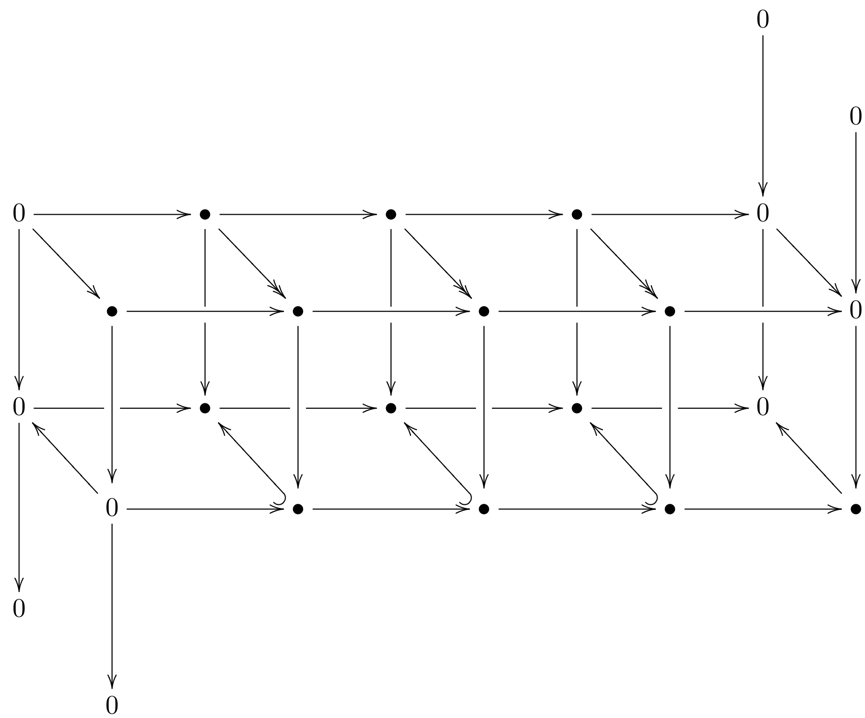
$$\kappa_n \colon Z^n(C^\bullet) \to \widetilde{Z}^{n+1}(A^\bullet).$$

It turns out the connecting maps c_n and κ_n are related. In fact we have the following:

Proposition 1.2.3. *The following diagram commutes*

$$\begin{tikzcd}
Z^n(C^\bullet) \arrow[r, "\kappa_n"] \arrow[d, two heads, "\varpi"'] & \widetilde{Z}^{n+1}(A^\bullet) \\
\mathrm{H}^n(C^\bullet) \arrow[r, "c_n"'] & \mathrm{H}^{n+1}(A^\bullet) \arrow[u, hook, "i"']
\end{tikzcd}$$

Proof. For any complex $X^\bullet$ we have functorial maps $X^n \twoheadrightarrow \widetilde{Z}^n(X^\bullet($ and $Z^{n+1}(X^\bullet) \hookrightarrow X^{n+1}$. It follows that the diagrams (1.2.1) and (1.2.2) are related. In fact we have a three dimensional commutative diagram whose skeleton is given below with the rear face arising from (1.2.1) and the front face from (1.2.2).



Let $(\bar{K},\, \bar{u}) = \ker(B^n \to C^{n+1})$ and $(\bar{K}',\, \bar{v}) = \operatorname{coker}(A^n \to B^{n+1})$ and $\bar{\varphi}\colon \bar{K} \to \bar{K}'$ the composite $\bar{\varphi} = \bar{v} \circ \partial^n_{B^\bullet} \circ \bar{u}$. Consider the following diagram:

(1.2.4)
$$\begin{tikzcd}
\bar{K} \arrow[rr, "\bar{\varphi}"] \arrow[dr] \arrow[dd] & & \bar{K}' & \\
& K \arrow[rr, "\varphi"] \arrow[dd] & & K' \arrow[ul] \\
Z^n(C^\bullet) \arrow[rr, "\kappa_n"] \arrow[dr, "\varpi"'] & & \widetilde{Z}^{n+1}(A^\bullet) \arrow[uu] & \\
& \mathrm{H}^n(C^\bullet) \arrow[rr, "c_n"'] & & \mathrm{H}^{n+1}(A^\bullet) \arrow[uu, hook] \arrow[ul, hook, "i"']
\end{tikzcd}$$

We have to show that the bottom face commutes. This is similar to the diagram in `notes7.pdf` (see proof of Lemma 1.1.2 in `notes7.pdf`). The orientation of two of the arrows are different. But as in that cube, it is easy to see that five faces other than the bottom face commute. Indeed, the front and rear faces commute by definition of c_n and κ_n. The west face commutes by the universal property of cokernels and kernels, and by duality the east face also commutes. The top face can be expanded as follows:

$$\begin{tikzcd}
\bar{K} \arrow[r] \arrow[d, "\bar{u}"'] \arrow[dr, phantom, "\square_1"] & K \arrow[d, "u"] \\
B^n \arrow[r] \arrow[d, "\partial^n"'] \arrow[dr, phantom, "\square_2"] & \widetilde{Z}^n(B^\bullet) \arrow[d, "\partial^n_*"] \\
B^{n+1} \arrow[d, "\bar{v}"'] \arrow[dr, phantom, "\square_3"] & Z^{n+1}(B^\bullet) \arrow[l] \arrow[d, "v"] \\
\bar{K}' & K' \arrow[l]
\end{tikzcd}$$

Since each of the sub-rectangles $\square_1$, $\square_2$, and $\square_3$ commutes, the outer rectangle also commutes. Thus the top face of (1.2.4) commutes. Now consider the diagram

$$\begin{tikzcd}
\bar{K} \arrow[r, two heads] & Z^n(C^\bullet) \arrow[r, "\kappa_n"] \arrow[d] & \widetilde{Z}^{n+1}(A^\bullet) \arrow[r, hook] & \bar{K}' \\
 & \mathrm{H}^n(C^\bullet) \arrow[r, "c_n"'] & \mathrm{H}^{n+1}(A^\bullet) \arrow[u, hook] &
\end{tikzcd}$$

From the comutativity of all except the bottom face of the cube (1.2.4), the two routes from $\bar{K}$ to $\bar{K}'$ are the same. Since $\bar{K} \twoheadrightarrow Z^n(C^\bullet)$ is an epimorphism and $\widetilde{Z}^{n+1}(A^\bullet) \hookrightarrow \bar{K}'$ is a monomorphism, the rectangle in the middle commutes. $\square$

2. Mapping Cones

Throughout this section $\mathscr{C}$ is an exact category and $\mathscr{A}$ is an abelian category.

2.1. Translations of complexes. Let $C^\bullet$ be complex in $\mathscr{C}$ and i and integer. Define a new complex $C^\bullet[i[$ in the following way.

$$(C^\bullet[i])^n := C^{n+i} \quad \text{and} \quad \partial^n_{C^\bullet[i]} = (-1)^i \partial^{n+i}_{C^\bullet}.$$

The complex $C^\bullet[i]$ is called the i-th translate of $C^\bullet$. Note that

$$\mathrm{H}^n(C^\bullet[i]) = \mathrm{H}^{n+i}(C^\bullet).$$

2.2. Mapping cones. Suppose

$$f\colon P^\bullet \to Q^\bullet$$

is map of complexes in $\mathscr{A}$ (recall, $\mathscr{A}$ is abelian).

Definition 2.2.1. The *mapping cone of* f, $C^\bullet = C^\bullet_f$ is the complex whose n-th graded piece is

$$C^n = P^{n+1} \oplus Q^n$$

and whose coboundary maps are given by the formula

$$\partial^n_C = \begin{pmatrix} -\partial^{n+1}_P & 0 \\ f^{n+1} & \partial^n_Q \end{pmatrix}$$

That ∂_C is indeed a coboundary (i.e., $C^\bullet$ is indeed a complex) is seen by the following computation:

$$\begin{aligned}
\partial_C^{n+1} \circ \partial_C^n &= \begin{pmatrix} -\partial_P^{n+2} & 0 \\ f^{n+2} & \partial_Q^{n+1} \end{pmatrix} \begin{pmatrix} -\partial_P^{n+1} & 0 \\ f^{n+1} & \partial_Q^{n} \end{pmatrix} \\
&= \begin{pmatrix} \partial_P^{n+1} \circ \partial_P^n & 0 \\ -f^{n+2} \circ \partial_P^{n+1} + \partial_Q^{n+1} \circ {}^{n+1} & \partial_Q^{n+1} \circ \partial_Q^n \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Consider the following diagram with split exact rows:

(2.2.2)
$$\begin{tikzcd}
0 \arrow[r] & Q^n \arrow[r, "\begin{pmatrix} 0 \\ 1_{Q^n} \end{pmatrix}"] \arrow[d, "\partial_Q^n"'] & P^{n+1} \oplus Q^n \arrow[r, "\begin{pmatrix} 1_{P^{n+1}} & 0 \end{pmatrix}"] \arrow[d, "\partial_C"'] & P[n+1] \arrow[r] \arrow[d, "\partial_P^{n+1}"] & 0 \\
0 \arrow[r] & P^{n+1} \arrow[r, "\begin{pmatrix} 0 \\ 1_{Q^{n+1}} \end{pmatrix}"'] & P^{n+2} \oplus Q^{n+1} \arrow[r, "\begin{pmatrix} 1_{P^{n+2}} & 0 \end{pmatrix}"'] & P^{n+2} \arrow[r] & 0
\end{tikzcd}$$

Using $\partial_C^n = \begin{pmatrix} -\partial_P^{n+1} & 0 \\ f^{n+1} & \partial_Q^n \end{pmatrix}$, we see that (2.2.2) commutes. Indeed, we have

$$\begin{pmatrix} -\partial_P^{n+1} & 0 \\ f^{n+1} & \partial_Q^n \end{pmatrix} \begin{pmatrix} 0 \\ 1_{Q^n} \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_Q^n \end{pmatrix} = \begin{pmatrix} 0 \\ 1_{Q^{n+1}} \end{pmatrix} \circ \partial_Q^n$$

and

$$\begin{pmatrix} 1_{P^{n+2}} & 0 \end{pmatrix} \begin{pmatrix} -\partial_P^{n+1} & 0 \\ f^{n+1} & \partial_Q^n \end{pmatrix} = \begin{pmatrix} \partial_P^{n+1} & 0 \end{pmatrix} = \partial_P^{n+1} \begin{pmatrix} 1_{P^{n+1}} & 0 \end{pmatrix}.$$

We therefore have a short exact-sequence of complexes

(2.2.3)
$$0 \to Q^\bullet \to C_f^\bullet \to P^\bullet[1] \to 0$$

Proposition 2.2.4. *Let $f\colon P^\bullet \to Q^\bullet$ be a map of complexes in $\mathscr{A}$. For $n \in \mathbb{Z}$ let $c_n\colon \mathrm{H}^{n+1}(P^\bullet) \to \mathrm{H}^{n+1}(Q^\bullet)$ be the connecting map arising from the short exact sequence of complexes* (2.2.3) *and the identity* $\mathrm{H}^n(P^\bullet[1]) = \mathrm{H}^{n+1}(P^\bullet)$. *Then*

$$c_n = \mathrm{H}^{n+1}(f).$$

Proof. As before, for any complex $X^\bullet$ and any integer n, $\varpi\colon Z^n(X^\bullet) \twoheadrightarrow \mathrm{H}^n(X^\bullet)$, $j\colon Z^n(X^\bullet) \hookrightarrow X^n$, $\pi\colon X^n \to \widetilde{Z}^n(X^\bullet$, and $i\colon \mathrm{H}^n(X^\bullet) \hookrightarrow \widetilde{Z}^n(X^\bullet)$ represent the canonical maps.

Fix $n \in \mathbb{Z}$. Recall from diagram (3.1.1) of `notes7.pdf` that we have a commutative diagram

$$
\begin{tikzcd}
P^{n+1} \arrow[r, "f^{n+1}"] & Q^{n+1} \arrow[d, two heads, "\pi"] \\
Z^{n+1}(P^\bullet) \arrow[u, hook, "j"] \arrow[d, two heads, "\varpi"'] & \widetilde{Z}^{n+1}(Q^\bullet) \\
\mathrm{H}^{n+1}(P^\bullet) \arrow[r, "\mathrm{H}^{n+1}(f)"'] & \mathrm{H}^{n+1}(Q^\bullet) \arrow[u, hook, "i"']
\end{tikzcd}
\tag{2.2.5}
$$

On the other hand, by Proposition 1.2.3 the following diagram commutes

$$
\begin{tikzcd}
Z^{n+1}(P^\bullet) \arrow[r, "\kappa_n"] \arrow[d, two heads, "\varpi"'] & \widetilde{Z}^{n+1}(Q^\bullet) \\
\mathrm{H}^{n+1}(P^\bullet) \arrow[r, "c_n"'] & \mathrm{H}^{n+1}(Q^\bullet) \arrow[u, hook, "i"']
\end{tikzcd}
\tag{2.2.6}
$$

Since ϖ is an epimorphism and i is a monomorphism, by (2.2.5) and (2.2.6) it is enough for us to show that

$$\kappa_n = \pi \circ f^{n+1} \circ j. \tag{2.2.7}$$

Now consider diagram (2.2.2). Let $(K,\, u) = \ker\left(P^{n+1} \oplus Q^n \to P^{n+2}\right)$ and $(K',\, v) = \operatorname{coker}\left(Q^n \to P^{n+2} \oplus Q^{n+1}\right)$. Now the map $P^{n+1} \oplus Q^n \to Q^{n+1}$ is $\begin{pmatrix} \partial_P^{n+1} & 0 \end{pmatrix}$, and $Q^n \to P^{n+2} \oplus Q^{n+1}$ is $\begin{pmatrix} 0 \\ \partial_Q^n \end{pmatrix}$. Hence by (1.1.1) and (1.1.2) we have

$$(K,\, u) = \left(Z^{n+1}(P^\bullet) \oplus Q^n, \begin{pmatrix} j & 0 \\ 0 & 1_{Q^n} \end{pmatrix} \right)$$

and

$$(K',\, v) = \left(P^{n+2} \oplus \widetilde{Z}^{n+1}(Q^\bullet), \begin{pmatrix} 1_{P^{n+2}} & 0 \\ 0 & \pi \end{pmatrix} \right).$$

Moreover, one checks easily that the two natural maps $K \twoheadrightarrow Z^n(P^\bullet[1]) = Z^{n+1}(P^\bullet)$ and $\widetilde{Z}^{n+1}(Q^\bullet) \hookrightarrow K'$ arising from (2.2.2) are $\begin{pmatrix} 1 & 0 \end{pmatrix} : Z^{n+1}(P^\bullet) \oplus Q^n \to Z^{n+1}(P^\bullet)$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} : \widetilde{Z}^{n+1}(Q^\bullet) \to P^{n+2} \oplus \widetilde{Z}^{n+1}(Q^\bullet)$ respectively.

By definition of κ_n the following diagram commutes:

$$
\begin{tikzcd}[column sep=large, row sep=huge]
 & P^{n+1} \oplus Q^n \arrow[r, "\begin{pmatrix} -\partial^{n+1} & 0 \\ f^{n+1} & \partial^n \end{pmatrix}"] & P^{n+2} \oplus Q^{n+1} \arrow[d, two heads, "\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}"] & \\
K \arrow[r, equal] & Z^{n+1}(P^\bullet) \oplus Q^n \arrow[u, hook, "\begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, two heads, "\begin{pmatrix} 1 & 0 \end{pmatrix}"'] & P^{n+2} \oplus \widetilde{Z}^{n+1}(Q^\bullet) \arrow[r, equal] & K' \\
 & Z^{n+1}(P^\bullet) \arrow[r, "\kappa_n"'] & \widetilde{Z}^{n+1}(Q^\bullet) \arrow[u, hook, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"'] &
\end{tikzcd}
$$

Thus

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} (\kappa_n) \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \begin{pmatrix} -\partial^{n+1} & 0 \\ f^{n+1} & \partial^n \end{pmatrix} \begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 0 & 0 \\ \kappa_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \pi \circ f^{n+1} \circ j & 0 \end{pmatrix}.$$

This gives (2.2.7), as required. □

Corollary 2.2.8. *One has a long exact sequence*

$$\dots \xrightarrow{\mathrm{H}^n(f)} \mathrm{H}^n(Q^\bullet) \longrightarrow \mathrm{H}^n(C_f^\bullet) \longrightarrow \mathrm{H}^{n+1}(P^\bullet) \xrightarrow{\mathrm{H}^{n+1}(f)} \mathrm{H}^{n+1}(Q^\bullet) \longrightarrow \dots$$

with the unlabelled maps arising from the exact sequence of complexes in (2.2.3).

Proof. This is clear from the Proposition. □

The main result of these notes on Mapping Cones is the following Corollary to Proposition 2.2.4.

Corollary 2.2.9. *A map of complexes in an abelian category is a quasi-isomorphism if and only if the corresponding mapping cone is exact.*

Proof. Immediate from Corollary 2.2.8 □