DIRECT SUMS AND PRODUCTS AND SPLIT EXACT SEQUENCES

1. **Direct sums and products in additive categories**

In this section $\mathscr{A}$ is an *additive category*

1.1. Let X and Y be objects in $\mathscr{A}$ and $i\colon x \to X \oplus Y$ and $j\colon Y \to X \oplus Y$ the universal maps (necessarily monomorphisms). By the universal property of direct sums, the maps $\mathbf{1}_X\colon X \to X$ and $0\colon Y \to X$ give us a unique map $p\colon X \oplus Y \to X$ such that $p \circ i = \mathbf{1_X}$ and $p \circ j = 0$. Similarly, we have a unique map $q\colon X \oplus Y \to Y$ characterised by $q \circ i = 0$ and $q \circ j = \mathbf{1}_Y$. Note that if $\phi \circ p = \psi \circ p$ then $\phi \circ p \circ i = \psi \circ p \circ i$, whence $\phi = \psi$ as $p \circ i = \mathbf{1}_X$. Thus p is an epimorphism. Similarly q is an epimorphism. We claim that

$$i \circ p + j \circ q = \mathbf{1}_{X \oplus Y}. \tag{1.1.1}$$

To see this, let $\varphi = i \circ p + j \circ q$. Then $\varphi \circ i = i \circ p \circ i + j \circ q \circ i = i = \mathbf{1}_{X \oplus Y} \circ i$. Similarly $\varphi \circ j = \mathbf{1}_{X \oplus Y} \circ j$, whence by the universal property of direct sums we have $\varphi = \mathbf{1}_{X \oplus Y}$, i.e., (1.1.1) is true.

Proposition 1.1.2. *The direct product of X and Y exists in $\mathscr{A}$. In fact $(X \oplus Y, p, q)$ is a direct product of X and Y.*

Proof. Let $x\colon T \to X$ and $y\colon T \to Y$ be two maps in $\mathscr{A}$. We have to show that there exists a unique map $f\colon T \to X \oplus Y$ such that $p \circ f = x$ and $q \circ f = y$. Uniqueness of a such an f follows from the fact that such an f must satisfy $f = i \circ x + j \circ y$. Indeed, if f is such that $p \circ f = x$ and $q \circ f = y$ then

$$\begin{aligned} f &= \mathbf{1}_{X \oplus Y} \circ f \\ &= (i \circ p + j \circ q) \circ f \qquad \text{(via (1.1.1))} \\ &= i \circ x + j \circ y. \end{aligned}$$

The existence of such an f is shown as follows. Let $f = i \circ x + j \circ y$. Then $p \circ f = x$ and $q \circ f = y$. □

Remarks 1.1.3. 1) Let us agree to call a category $\mathscr{A}$ a *pre-additive* category if $\mathscr{A}$ has a zero object 0, for every pair of objects X and Y in $\mathscr{A}$ we have $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is an abelian group with the zero map as the zero element, and for $f, g \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$ and maps $\sigma\colon W \to X$ and $\tau\colon Y \to Z$ in $\mathscr{A}$ we have $(f + g) \circ \sigma = f \circ \sigma + g \circ \sigma$ and $\tau \circ (f + g) = \tau \circ f + \tau \circ g$. Then the dual of Proposition 1.1.2 is that if a pre-additive category $\mathscr{A}$ has direct products then it has direct sums. In other words the following are equivalent for a pre-additive category $\mathscr{A}$:

- $\mathscr{A}$ had direct sums.
- $\mathscr{A}$ has direct products.
- $\mathscr{A}$ is an additive category.

2) From 1) it follows that the opposite category of an additive category is also an additive category.

3) Since the opposite category of an exact category is also an exact category, from 2) we get that the opposite category of an abelian category is an abelian category.

1.2. The matrix notation. Let $\mathscr{A}$ be an additive category. Let X_j, $j = 1, \dots, n$ and Y_i, $i = 1, \dots, m$ be objects in $\mathscr{A}$. Set $X = \bigoplus_{j=1}^n X_j$ and $Y = \bigoplus_{i=1}^m Y_i$. Since Y is also the direct product of the Y_i's by Proposition 1.1.2, we have epimorphisms $p_i \colon Y \to Y_i$. Moroever a map $y \colon T \to Y$ is characterised by maps $y_i \colon T \to Y_i$, $i = 1, \dots, m$, such that $y_i = p_i \circ y$ for every i. Let us agree to denote the map y by the column vector whose i-th entry is y_i, i.e.,

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

Since X is a direct sum, any map $z \colon X \to Z$ is completely determined by maps $z_j \colon X_j \to Z$, $j = 1, \dots, n$ where z_j are the composites $X_j \hookrightarrow X \xrightarrow{z} Z$ for $j = 1, \dots, n$, and where the monomorphism $X_j \hookrightarrow X$ is the universal map. Let us agree to denote z by a row vector whose j-th entry is x_j. Thus

$$z = \begin{pmatrix} z_1 & \dots & z_n \end{pmatrix}$$

It follows that any map $\varphi \colon X \to Y$ can be written as an $m \times n$ matrix

$$\varphi = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & & \vdots \\ \varphi_{m1} & \cdots & \varphi_{mn} \end{pmatrix}$$

with the j-th column representing the map $X_j \hookrightarrow X \xrightarrow{\varphi} Y$, or equivalently, the i-th row representing the map $X \xrightarrow{\varphi} Y \xrightarrow{p_i} Y_i$. In other words, for $1 \leq i \leq m$ and $1 \leq j \leq n$ we have $\varphi_{ij} \colon X_j \to Y_i$ given by the composite $X_j \hookrightarrow X \xrightarrow{\varphi} Y \xrightarrow{p_i} Y_i$. If $Z = \bigoplus_{h=1}^l Z_i$ and $\psi \colon Y \to Z$ is a map, say

$$\psi = \begin{pmatrix} \psi_{11} & \dots & \psi_{1m} \\ \vdots & & \vdots \\ \psi_{l1} & \dots & \psi_{lm} \end{pmatrix}$$

then

$$\psi \circ \varphi = \begin{pmatrix} \psi_{11} & \dots & \psi_{1m} \\ \vdots & & \vdots \\ \psi_{l1} & \dots & \psi_{lm} \end{pmatrix} \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & & \vdots \\ \varphi_{m1} & \cdots & \varphi_{mn} \end{pmatrix}$$

where the right side is the usual matrix product.

2. Split exact sequences

Let $\mathscr{A}$ be an abelian category and A, B objects in $\mathscr{A}$. Let $i \colon A \to A \oplus B$, $j \colon B \to A \oplus B$, $p \colon A \oplus B \to A$, $q \colon A \oplus B \to B$ be the canonical maps arising from the role of $A \oplus B$ as a direct sum as well as a direct product. It is easy to see that

$$0 \to A \xrightarrow{i} A \oplus B \xrightarrow{q} B \to 0$$

is an exact sequence. In greater detail, first note $p \circ i = 1_A$, $q \circ i = 0$, $p \circ j = 0$ and $q \circ j = 1_B$. Moreover, $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $p = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Suppose $f\colon T \to A \oplus B$ is a map in $\mathscr{A}$. Then $f = \begin{pmatrix} a \\ b \end{pmatrix}$, where $a\colon T \to A$ and $b\colon T \to B$ are $a = p \circ f$ and $b = q \circ f$. Suppose further that $q \circ f = 0$. Then $b = 0$, whence

$$f = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} a = i \circ a.$$

Since i is a monomorphism, $x = a$ is the only solution of the equation $f = i \circ x$. It follows that $(A, i) = \ker q$. Since q is an epimorphism, the displayed sequence is exact. The sequence is an example of a split exact sequence, about which we say more in the following subsection.

2.1. Let

$$(*) \qquad 0 \to A \xrightarrow{s} X \xrightarrow{\pi} B \to 0$$

be a sequence of maps in the abelian category $\mathscr{A}$ and let i, j, p, q be as before.

Proposition 2.1.1. *The following are equivalent for the sequence* $(*)$.

(1) *There is an isomorphism* $\psi\colon X \xrightarrow{\sim} A \oplus B$ *such that the diagram below commutes:*

$$\begin{CD} 0 @>>> A @>{s}>> X @>{\pi}>> B @>>> 0 \\ @. @| @V{\wr}V{\psi}V @| @. \\ 0 @>>> A @>>{i}> A \oplus B @>>{q}> B @>>> 0 \end{CD}$$

(2) *The sequence* $(*)$ *is exact and there is a map* $\tau\colon X \to A$ *such that* $\tau \circ s = 1_A$.

(3) *The sequence* $(*)$ *is exact and there is a map* $\sigma\colon B \to X$ *such that* $\pi \circ \sigma = 1_B$.

(4) *There exist maps* $\tau\colon X \to A$ *and* $\sigma\colon B \to X$ *satisfying the three relations (a)* $\tau \circ s = 1_A$*, (b)* $\pi \circ \sigma = 1_B$*, and (c)* $s \circ \tau + \sigma \circ \pi = 1_X$.

Proof. It is clear that (1) implies (2). Suppose (2) is true. Consider the commutative diagram with exact rows

$$\begin{CD} 0 @>>> A @>{s}>> X @>{\pi}>> C @>>> 0 \\ @. @| @V{\tau}VV @VVV @. \\ 0 @>>> A @>>{1_A}> A @>>> 0 @>>> 0 \end{CD}$$

By the Snake Lemma we get that $\ker \tau \xrightarrow{\sim} C$. In other words we have map $\sigma\colon C \to X$ such that $(C, \sigma) = \ker \tau$. We have a commutative diagram

$$\begin{CD} C @= C \\ @V{\sigma}VV @| \\ X @>>{\pi}> C \end{CD}$$

This proves (3). More can be said about the relationship of the σ just found to τ, s and π. First note that $\tau \circ (1_X - s \circ \tau) = \tau - \tau \circ s \circ \tau = \tau - 1_A \circ \tau = 0$. Since $(C, \sigma) =$

$\ker \tau$, this means there is unique map $\pi' \colon X \to C$ such that $\sigma \circ \pi' = 1_X - s \circ \tau$. We then have the following sequence of equalities:

$$\begin{aligned} \pi' &= \pi \circ \sigma \circ \pi' \\ &= \pi \circ (1_X - s \circ \tau) \\ &= \pi - \pi \circ s \circ \tau \\ &= \pi \end{aligned}$$

since $\pi \circ s = 0$. This means $\sigma \circ \pi = 1_X - s \circ \tau$. Thus (2) implies that if σ is defined to be the kernel of τ, then σ satisfies $\pi \circ \sigma = 1_B$ and $s \circ \tau + \sigma \circ \pi = 1_X$. Thus (2) implies (4) also.

By duality we see that (3) implies (2) with $\tau = \operatorname{coker} \sigma$ and again by duality that it also implies (4). Thus (2) and (3) are equivalent statements with τ of (2) related to σ of (3) by $\sigma = \ker \tau$ and $\tau = \operatorname{coker} \sigma$ and when so related, we have $s \circ \tau + \sigma \circ \pi = 1_X$.

Finally suppose (4) is true. The relations $\tau \circ s = 1_A$ and $\pi \circ \sigma = 1_B$ imply that π and τ are epimorphisms and s and σ are monomorphisms. Then $\pi = \pi \circ (s \circ \tau + \sigma \circ \pi) = \pi \circ s \circ \tau + \pi$. Hence $\pi \circ s \circ \tau = 0$. Since τ is an epimorphism, we get $\pi \circ s = 0$. By duality we get $\tau \circ \sigma = 0$. Define $\psi \colon X \to A \oplus B$ by the formula

$$\psi = \begin{pmatrix} \tau \\ \pi \end{pmatrix}.$$

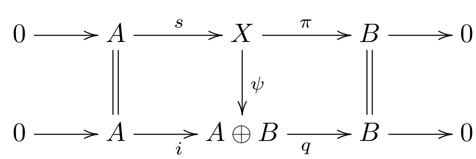
It follows that

$$\psi \circ s = \begin{pmatrix} \tau \\ \pi \end{pmatrix} s = \begin{pmatrix} \tau \circ s \\ \pi \circ s \end{pmatrix} = \begin{pmatrix} 1_A \\ 0 \end{pmatrix} = i$$

and since $q = \begin{pmatrix} 0 & 1_B \end{pmatrix}$, we have

$$q \circ \psi = \begin{pmatrix} 0 & 1_B \end{pmatrix} \begin{pmatrix} \tau \\ \pi \end{pmatrix} = \pi.$$

Thus the diagram



commutes. It remains to show that ψ is an isomorphism. Define $\varphi \colon A \oplus B \to X$ by the formula $\varphi = \begin{pmatrix} s & \sigma \end{pmatrix}$. Then

$$\varphi \circ \psi = \begin{pmatrix} s & \sigma \end{pmatrix} \begin{pmatrix} \tau \\ \pi \end{pmatrix} = s \circ \tau + \sigma \circ \pi = 1_X$$

by the hypothesis in (4). On the other hand, since we have shown $\pi \circ s = \tau \circ \sigma = 0$, we have

$$\psi \circ \varphi = \begin{pmatrix} \tau \\ \pi \end{pmatrix} \begin{pmatrix} s & \sigma \end{pmatrix} = \begin{pmatrix} \tau \circ s & \tau \circ \sigma \\ \pi \circ s & \pi \circ \sigma \end{pmatrix} = \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix} = 1_{A \oplus B}.$$

Thus ψ is an isomorphism. This gives (1). □

Definition 2.1.2. The sequence $(*)$ is said to be *split exact* if it satisfies any of the equivalent conditions of Proposition 2.1.1.