COHOMOLOGY OF COMPLEXES

1. **Snake Lemma again**

Let $\mathscr{C}$ be an exact category.

1.1. **Functoriality of the Snake Lemma.** Suppose we are given two commutative diagrams, $(*)_1$ and $(*)_2$ with exact rows and columns as below.

$(*)_k$

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
A_k \arrow[r] \arrow[d, "\alpha_k"'] & B_k \arrow[r] \arrow[d, "\beta_k"'] & C_k \arrow[r] \arrow[d, "\gamma_k"] & D_k \arrow[r] \arrow[d, "\delta_k"] & E_k \arrow[d, "\epsilon_k"] \\
A'_k \arrow[r] \arrow[d] & B'_k \arrow[r] & C'_k \arrow[r] & D'_k \arrow[r] & E'_k \\
0 & & & &
\end{tikzcd}$$

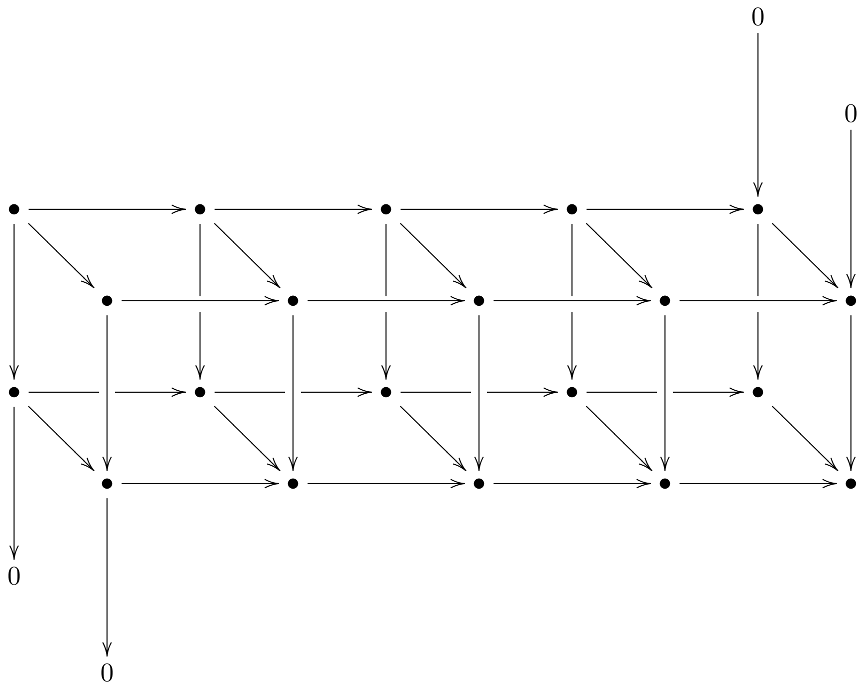
Fix $k \in \{1,2\}$. Let $K_k = \ker(C_k \to D'_k)$ and $K'_k = \operatorname{coker}(B_k \to C'_k)$ and $\varphi_k \colon K_k \to K'_k$ the composite

$$K_k \hookrightarrow C_k \xrightarrow{\gamma_k} C'_k \twoheadrightarrow K'_k.$$

If $\kappa_k \colon \ker \delta_k \to \operatorname{coker} \beta_k$ is the connecting homomorphism given by the Snake Lemma, then we know that κ_k fits into the commutative diagram

(1.1.1)
$$\begin{tikzcd}
K_k \arrow[r, "\varphi_k"] \arrow[d] & K'_k \\
\ker \delta_k \arrow[r, "\kappa_k"'] & \operatorname{coker} \beta_k \arrow[u, hook]
\end{tikzcd}$$

Next suppose we have a "map" $(*)_1 \to (*)_2$ between the two commutative diagrams, i.e., a set of maps $A_1 \to A_2$, $B_1 \to B_2$, $\ldots$,$E_1 \to E_2$, $A'_1 \to A'_2$, $\ldots E'_1 \to E'_2$ such that the resulting three dimensional diagram commutes. The skeleton of the diagram is given below with the front face representing $(*)_2$ and the rear face representing $(*)_1$.



Lemma 1.1.2. *In the above situation the following diagram commutes*

$$\begin{tikzcd}
\ker\delta_1 \arrow[r, "\kappa_1"] \arrow[d] & \operatorname{coker}\beta_1 \arrow[d] \\
\ker\delta_2 \arrow[r, "\kappa_2"'] & \operatorname{coker}\beta_2
\end{tikzcd}$$

where the unlabelled arrows are the natural ones.

Remark: This is a way of saying that the connecting map in the Snake Lemma is "functorial".

Proof. We only sketch the proof and leave it to the reader to flesh out the details. Consider the following cube:

$$\begin{tikzcd}
K_1 \arrow[rr, "\varphi_1"] \arrow[dr] \arrow[dd] & & K_1' \arrow[dr] & \\
& K_2 \arrow[rr, "\varphi_2"] \arrow[dd] & & K_2' \\
\ker\delta_1 \arrow[rr, "\kappa_1"] \arrow[dr] & & \operatorname{coker}\beta_1 \arrow[uu] \arrow[dr] & \\
& \ker\delta_2 \arrow[rr, "\kappa_2"'] & & \operatorname{coker}\beta_2 \arrow[uu]
\end{tikzcd}$$

We have to show that the bottom face commutes. It is easy to see the remaining five faces commute. For example the front and rear faces commute by (1.1.1). The west face commutes by universal properties of kernels, and the east face by the universal properties of cokernels. The top face commutes because the following three rectangles commute, namely (a) the rectangle with vertices K_1, K_2, C_1, and C_2; (b) the rectangle with vertices C_1, C_2, C_1', C_2'; and (c) the rectangle with vertices C_1', C_2', K_1', and K_2'. To prove that the bottom face of our cube commutes, consider the diagram:

$$\begin{tikzcd}
K_1 \arrow[dr] & & & \\
& \ker\delta_1 \arrow[r, "\kappa_1"] \arrow[d] & \operatorname{coker}\beta_1 \arrow[d] & \\
& \ker\delta_2 \arrow[r, "\kappa_2"'] & \operatorname{coker}\beta_2 \arrow[dr, hook] & \\
& & & K_2'
\end{tikzcd}$$

From our earlier analysis of the cube above this diagram, the two routes from K_1 to K_2' presented by the diagram are the same. Since the diagonal arrow starting at the northwest corner is an epimorphism and the diagonal arrow ending at the southeast corner is a monomorphism, it follows that the rectangle in the middle commutes. □

2. **Cohomology**

In this section $\mathscr{C}$ is an *exact category.*

2.1. **Notations.** 1) If $i\colon A \to B$ is a *monomorphism* in $\mathscr{C}$ then we often write $(B/A, p)$ for the cokernel of i. Thus B/A makes sense as an object (up to isomorphism) whenever A is a sub-object of B. If the epimorphism p is remembered, then $(B/A, p)$ is unique up to unique isomorphism.

2) If $(A^\bullet, \partial)$ is a complex of objects in $\mathscr{A}$, we write $Z^n(A^\bullet) = \ker(A^n \xrightarrow{\partial^n} A^{n+1})$, $\widetilde{Z}^n(A^\bullet) = \operatorname{coker}(A^{n-1} \xrightarrow{\partial^{n-1}} A^n)$, and $B^n(A^\bullet) = \mathbf{im}(\partial^{n-1})$. When the context is clear, we write Z^n, $\widetilde{Z}^n$, B^n instead of $Z^n(A^\bullet)$, etc.

2.2. **Cohomology.** Let $A^\bullet$ be a complex of objects in $\mathscr{C}$. Recall that the n-th cohomology of $A^\bullet$ (for $n \in \mathbb{Z}$) is

$$\mathrm{H}^n(A^\bullet) = Z^n(A^\bullet)/B^n(A^\bullet).$$

There are other descriptions of $\mathrm{H}^n(A^\bullet)$. Since the composite $A^{n-1} \to A^n \to A^{n+1}$ is zero, by definition of a cokernel, we have a map $\widetilde{Z}^n \to A^n$. Moreover, the composite $\widetilde{Z}^n \to A^{n+1} \to A^{n+2}$ is zero and hence by definition of a kernel, the map $\widetilde{Z}^n \to A^{n+1}$ factors uniquely as $\widetilde{Z}^n \to Z^{n+1} \hookrightarrow A^{n+1}$. Thus we have maps

$$\partial^n_*\colon \widetilde{Z}^n \to Z^{n+1} \qquad (n \in \mathbb{Z}) \tag{2.2.1}$$

which fit into a commutative diagrams (one for each $n \in \mathbb{Z}$)

$$\begin{tikzcd} A^n \arrow[r, "\partial^n"] \arrow[d, two heads] & A^{n+1} \\ \widetilde{Z}^n \arrow[r, "\partial^n_*"'] & Z^{n+1} \arrow[u, hook] \end{tikzcd}$$

It is clear that

(2.2.2) $$\mathrm{H}^n(A^\bullet) = \ker \partial^n_* \quad \text{and} \quad \mathrm{H}^{n+1}(A^\bullet) = \operatorname{coker} \partial^n_* \qquad (n \in \mathbb{Z}).$$

Next let $f\colon A^\bullet \to B^\bullet$ be a map of complexes. For each $n \in \mathbb{Z}$ have a natural map $Z^n(A^\bullet) \to Z^n(B^\bullet)$ arising from the universal property of kernels. We therefore have the composite $Z^n(A^\bullet) \to Z^n(B^\bullet) \twoheadrightarrow \mathrm{H}^n(B^\bullet)$. It is clear that the composite $B^n(A^\bullet) \hookrightarrow Z^n(A^\bullet) \to \mathrm{H}^n(B^\bullet)$ is zero. Hence we get a map

(2.2.3) $$\mathrm{H}^n(f)\colon \mathrm{H}^n(A^\bullet) \to \mathrm{H}^n(B^\bullet).$$

One checks easily that $\mathrm{H}^n(f \circ g) = \mathrm{H}^n(f) \circ \mathrm{H}^n(g)$. In other words, H^n is a functor from the category of complexes on $\mathscr{C}$ to $\mathscr{C}$.

Theorem 2.2.4. *Let*

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$$

be an exact sequence of complexes in the exact category $\mathscr{C}$. Then there exist maps $c_n\colon \mathrm{H}^n(C^\bullet) \to \mathrm{H}^n(A^\bullet)$, one for each $n \in \mathbb{Z}$, giving an exact sequence

$$\dots \xrightarrow{\mathrm{H}^n(f)} \mathrm{H}^n(B^\bullet) \xrightarrow{\mathrm{H}^n(g)} \mathrm{H}^n(C^\bullet) \xrightarrow{c_n} \mathrm{H}^{n+1}(A^\bullet) \xrightarrow{\mathrm{H}^{n+1}(f)} \mathrm{H}^{n+1}(B^\bullet) \xrightarrow{\mathrm{H}^{n+1}(g)} \dots$$

Moreover if we have a commutative diagram of complexes with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A_1^\bullet \arrow[r, "f_1"] \arrow[d, "a"'] & B_1^\bullet \arrow[r, "g_1"] \arrow[d, "b"'] & C_1^\bullet \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A_2^\bullet \arrow[r, "f_2"'] & B_2^\bullet \arrow[r, "g_2"'] & C_2^\bullet \arrow[r] & 0 \end{tikzcd}$$

then using c_n as the symbol for the n-th connecting map for both the short exact sequences in the diagram, the following diagram commutes for every $n \in \mathbb{Z}$:

$$\begin{tikzcd} \mathrm{H}^n(C_1^\bullet) \arrow[r, "c_n"] \arrow[d, "\mathrm{H}^n(c)"'] & \mathrm{H}^n(A_1^\bullet) \arrow[d, "\mathrm{H}^n(a)"] \\ \mathrm{H}^n(C_2^\bullet) \arrow[r, "c_n"'] & \mathrm{H}^n(A_2^\bullet) \end{tikzcd}$$

Proof. For $n \in \mathbb{Z}$ consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A^n \arrow[r, "f^n"] \arrow[d, "\partial^n"'] & B^n \arrow[r, "g^n"] \arrow[d, "\partial^n"'] & C^n \arrow[r] \arrow[d, "\partial^n"] & 0 \\ 0 \arrow[r] & A^{n+1} \arrow[r, "f^{n+1}"'] & B^{n+1} \arrow[r, "g^{n+1}"'] & C^{n+1} \arrow[r] & 0 \end{tikzcd}$$

By the Snake Lemma, for each n we have two exact sequences

$$0 \to Z^n(A^\bullet) \to Z^n(B^\bullet) \to Z^n(C^\bullet)$$

and

$$\widetilde{Z}^{n+1}(A^\bullet) \to \widetilde{Z}^{n+1}(B^\bullet) \to \widetilde{Z}^{n+1}(C^\bullet) \to 0.$$

We therefore have, for each n, a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
\widetilde{F} \arrow[r] \arrow[d] & \widetilde{Z}^n(A^\bullet) \arrow[r] \arrow[d, "\partial_*^n"'] & \widetilde{Z}^n(B^\bullet) \arrow[r] \arrow[d, "\partial_*^n"'] & \widetilde{Z}^n(C^\bullet) \arrow[r] \arrow[d, "\partial_*^n"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & Z^{n+1}(A^\bullet) \arrow[r] & Z^{n+1}(B^\bullet) \arrow[r] & Z^{n+1}(C^\bullet) \arrow[r] & F \\
0 & & & &
\end{tikzcd}$$

where the maps ∂_*^n are as in (2.2.1), $\widetilde{F} = \ker(\widetilde{Z}^n(A^\bullet) \to \widetilde{Z}^n(B^\bullet))$, and $F = \operatorname{coker}(Z^n(B^\bullet) \to Z^n(C^\bullet))$. Applying the Snake Lemma and using (2.2.2) we get the required long exact sequence. The second assertion of the theorem is an immediate consequence of Lemma 1.1.2. $\square$

3. **Homotopy**

3.1. **The functor $A^\bullet \mapsto \mathrm{H}^n(A^\bullet)$ revisited.** Suppose $\mathscr{C}$ is an exact category and $A^\bullet$ is a complex in $\mathscr{C}$. Let $Z^n = Z^n(A^\bullet)$ and $\widetilde{Z}^n = \widetilde{Z}(A^\bullet)$ be as before. We have a commutative diagram

$$\begin{tikzcd}
Z^n \arrow[r, hook, "j"] \arrow[d, two heads, "\varpi"'] & A^n \arrow[d, two heads, "\pi"] \\
\mathrm{H}^n(A^\bullet) \arrow[r, hook, "i"'] & \widetilde{Z}^n
\end{tikzcd}$$

where the horizontal arrows are monomorphisms and the downward arrows are epimorphisms.

Now suppose $f\colon A^\bullet \to B^\bullet$ is a map pf complexes. It is easy to see that we have a commutative cube:

$$\begin{tikzcd}
A^n \arrow[rr, "f^n"] \arrow[dd, two heads, "\pi"'] & & B^n \arrow[dd, two heads, "\pi"] & \\
 & Z^n(A^\bullet) \arrow[ul, hook, "j"'] \arrow[rr] \arrow[dd, two heads, "\varpi"] & & Z^n(B^\bullet) \arrow[ul, hook, "j"'] \arrow[dd, two heads, "\varpi"] \\
\widetilde{Z}^n(A^\bullet) \arrow[rr] & & \widetilde{Z}^n(B^\bullet) & \\
 & \mathrm{H}^n(A^\bullet) \arrow[ul, hook, "i"] \arrow[rr, "\mathrm{H}^n(f)"'] & & \mathrm{H}^n(B^\bullet) \arrow[ul, hook, "i"]
\end{tikzcd}$$

In particular we have the following commutative diagram

$$
\begin{tikzcd}
A^n \arrow[r, "f^n"] & B^n \arrow[d, two heads, "\pi"] \\
Z^n(A^\bullet) \arrow[u, hook, "j"] \arrow[d, two heads, "\varpi"'] & \widetilde{Z}^n(B^\bullet) \\
\mathrm{H}^n(A^\bullet) \arrow[r, "\mathrm{H}^n(f)"'] & \mathrm{H}^n(B^\bullet) \arrow[u, hook, "i"']
\end{tikzcd}
\tag{3.1.1}
$$

3.2. **Homotopies.** Throughout this subsection all objects and all maps are in an *abelian category* $\mathscr{A}$.

Definition 3.2.1. Let $f, g\colon A^\bullet \rightrightarrows B^\bullet$ be two maps of complexes in $\mathscr{A}$. A *homotopy from f to g* is a sequence $s = (s^n)_{n\in\mathbb{Z}}$ of maps $s^n\colon A^n \to B^{n-1}$ such that

$$f^n - g^n = \partial^{n-1}_{B^\bullet} s^n + s^{n+1}\partial^n_{A^\bullet} \qquad (n \in \mathbb{Z}).$$

If this is so, we write $f \sim g$. Note that $f \sim g$ if and only if $g \sim f$. We say f is *homotopic* to g if $f \sim g$.

Proposition 3.2.2. *Let $f, g\colon A^\bullet \rightrightarrows B^\bullet$ be two maps of complexes in $\mathscr{A}$ such that $f \sim g$. Then*

$$\mathrm{H}^n(f) = \mathrm{H}^n(g) \qquad (n \in \mathbb{Z}).$$

Proof. Fix $n \in \mathbb{Z}$. Let i, j, π, and ϖ be as in the previous subsection. Note that $\partial^n_{A^\bullet} \circ j = 0$ and $\pi \circ \partial^{n-1}_{B^\bullet} = 0$. It follows that

$$\pi \circ (f^n - g^n) \circ j = \pi \circ (\partial^{n-1}_{B^\bullet} s^n + s^{n+1}\partial^n_{A^\bullet}) \circ j = 0.$$

Thus $\pi \circ f^n \circ j = \pi \circ g^n \circ j$. According to the commutative diagram (3.1.1), this means

$$i \circ \mathrm{H}^n(f) \circ \varpi = i \circ \mathrm{H}^n(g) \circ \varpi.$$

Since i is a monomorphism and ϖ is an epimorphism, this means $\mathrm{H}^n(f) = \mathrm{H}^n(g)$.
$\square$