# THE SNAKE LEMMA

## 1. **Preliminaries**

**1.1.** Recall from Problem 4 of HW 5 that if we have a commutative diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\alpha"] & Q \arrow[r] \arrow[d, "\beta"] & R \arrow[r] \arrow[d, "\gamma"] & S \arrow[d, "\delta"] \\
P' \arrow[r] \arrow[d] & Q' \arrow[r] & R' \arrow[r] & S' \\
0 & & &
\end{tikzcd}$$

such that the rows are exact and the column on the left is exact then the the sequence

$$\ker \beta \to \ker \gamma \to \ker \delta$$

is exact. The dual of the statement is that if

(∗)

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
P \arrow[r] \arrow[d, "\alpha"] & Q \arrow[r] \arrow[d, "\beta"] & R \arrow[r] \arrow[d, "\gamma"] & S \arrow[d, "\delta"] \\
P' \arrow[r] & Q' \arrow[r] & R' \arrow[r] & S'
\end{tikzcd}$$

is a commutative diagram with exact rows and with the column on the right also exact, then the sequence

$$\text{coker}\, \alpha \to \text{coker}\, \beta \to \text{coker}\, \gamma \qquad (**)$$

is exact.

**1.2. The Statement.** Our version of the snake lemma (a somewhat non-canonical version) is as follows:

**Theorem 1.2.1** (The Snake Lemma)**.** *Suppose we have a commutative diagram with exact rows and columns:*

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r] \arrow[d] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \\
0 & & & &
\end{tikzcd}$$

*Then we have a six term exact sequence*

$$\ker\beta \to \ker\gamma \to \ker\delta \xrightarrow{\partial} \operatorname{coker}\beta \to \operatorname{coker}\gamma \to \operatorname{coker}\delta$$

*where the "connecting homomorphism" $\partial\colon \ker\delta \to \operatorname{coker}\beta$ is characterised as follows:*

(1) *Let $K = \ker(C \to D')$. Then the natural map $\pi\colon K \to \ker\delta$ is an epimorphism.*

(2) *Let $K' = \operatorname{coker}(B \to C')$. Then the natural map $j\colon \operatorname{coker}\beta \to K'$ is a monomorphism.*

(3) *The following diagram commutes*

$$\begin{tikzcd}
C \arrow[r, "\gamma"] & C' \arrow[d, two heads] \\
K \arrow[u, hook] \arrow[d, "\pi"', two heads] & K' \\
\ker\delta \arrow[r, "\partial"'] & \operatorname{coker}\beta \arrow[u, hook, "j"']
\end{tikzcd}$$

**Remarks 1.2.2.** In what follows let $\lambda\colon K \to K'$ be the composite $K \hookrightarrow C \xrightarrow{\gamma} C' \twoheadrightarrow K'$.

(a) Statements (1), (2), and (3) do indeed characterise $\partial$. Indeed, via these statements we have a commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\lambda"] \arrow[d, "\pi"', two heads] & K' \\
\ker\delta \arrow[r, "\partial"'] & \operatorname{coker}\beta \arrow[u, hook, "j"']
\end{tikzcd}$$

If $\partial'\colon \ker\delta \to \operatorname{coker}\beta$ is another map satisfying (3), and if (1) and (2) are true, then

$$j \circ \partial \circ \pi = j \circ \partial' \circ \pi,$$

for both sides equal $\lambda$. Since $\pi$ is an epimorphism and $j$ is a monomorphism, they can be cancelled from the right and left respectively, and we get $\partial = \partial'$.

(b) Suppose all objects were abelian groups and all maps group homomorphisms in the above. More precisely, suppose our category $\mathscr{C}$ was an exact subcategory of the category of abelian groups such that kernels and cokernels of maps in $\mathscr{C}$ agree with the kernels and cokernels in the category of abelian groups. Then the connecting map has the following description.

Let $d \in \ker\delta$. Then $d$ maps to 0 under $D \to E'$. Now $D \to E'$ factors as $D \to E \xrightarrow{\epsilon} E'$, and since $\epsilon$ is a monomorphism, $d \in \ker(D \to E)$. It follows that there exists an element $c \in C$ such that $c$ maps to $d$ under $C \to D$. Let $c' = \gamma(c)$. Clearly the image of $c'$ in $D'$ is 0 (for, under $C' \to D'$, we have $c' \mapsto \delta(d) = 0$). There is therefore an element $b' \in B'$ such that $b' \mapsto c'$ under $B' \to C'$. Let $[b']\,\mathrm{coker}\,\beta$ denote the image of $b'$ under $B' \twoheadrightarrow \mathrm{coker}\,\beta$. One checks readily that $[b']$ does not depend on the choices made, i.e., on the choice of the preimage $c$ of $d$, and the choice of the primage $b'$ of $c' = \gamma(c)$. Then we claim

$$\partial(d) = [b'].$$

In fact, the element $c$ lies in $K = \ker(C \to D')$ and the image of $\gamma(c)$ in $K' = \mathrm{coker}(B \to C')$ is precisely the image of $[b']$ in $K'$ under the natural map $\mathrm{coker}\,\beta \to K'$. Thus the map $\partial'\colon \ker\delta \to \mathrm{coker}\,\beta$ given by $d \mapsto [b']$ satisfies $j \circ \partial' \circ \pi = \lambda$ with $j$ and $\pi$ as before. It follows that $\partial' = \partial$ by our previous remark.

## 2. The Proof

**2.1.** By Problem 4 of HW 5 and its dual (see $(*)$ and $(**)$ of §§ 1.1 of these notes) we already have two exact sequences, namely

$$\ker\beta \to \ker\gamma \to \ker\delta \tag{2.1.1}$$

and

$$\mathrm{coker}\,\beta \to \mathrm{coker}\,\gamma \to \mathrm{coker}\,\delta \tag{2.1.2}$$

Let us prove (1) and (2) in the statement of the Lemma. Let the natural map $C' \to \ker(D' \to E')$ be denoted $f$. Note that $K = \ker(C \to \mathbf{im}(f))$. Consider the exact commutative diagram (i.e., all rows and columns of the diagram are commutative).

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f\circ\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbf{im}(f) \arrow[r] & D' \arrow[r] & E' \\
0 & 0 & & &
\end{tikzcd}$$

Apply Problem 4 of HW-5 twice to conclude that the resulting sequence of kernels

$$B \to K \to \ker\delta \to 0 \tag{2.1.3}$$

is exact. This proves (1). Statement (2) is the dual of (1). If you wish you can reprove it as follows. Let $g\colon \mathrm{coker}(A \to B) \to C$ be the natural map arising from the fact that the composite $A \to B \to C$ is 0, and let $i\colon \mathbf{im}(g) \to C$ be the natural monomorphism. Consider the following exact commutative diagram:

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] \\
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & \mathbf{im}\,(g) \arrow[r] \arrow[d, "g \circ i"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}
$$

Applying $(**)$ of §§ 1.1 twice to the above (with two different realisations of $(*)$), we get an exact sequence

$$(2.1.4) \qquad 0 \to \operatorname{coker}\beta \to K' \to D'.$$

This proves (2) of the Snake Lemma. As before let $\lambda\colon K \to K'$ be the composite $K \hookrightarrow C \xrightarrow{\gamma} C' \twoheadrightarrow K'$. Then the diagram

$$
\begin{tikzcd}
B \arrow[r] \arrow[d] & K \arrow[d, "\lambda"] \\
C' \arrow[r] & K'
\end{tikzcd}
$$

commutes where the horizontal arrow at the bottom arises because $K' = \operatorname{coker}(B \to C')$. The map obtained by starting from the northwest corner of the diagram, moving south and then east is clearly zero since $K' = \operatorname{coker}(B \to C')$. By (2.1.3), $\ker\delta = \operatorname{coker}(B \to K)$ whence we get a map

$$(2.1.5) \qquad \ker\delta \to K',$$

namely the unique map such that $\lambda\colon K \to K'$ equals the composite

$$K \xrightarrow{\text{via } (2.1.3)} \ker\delta \xrightarrow{(2.1.5)} K'.$$

Now $K \xrightarrow{\lambda} K' \to D'$ is zero, and by (2.1.3), $K \to \ker\delta$ is an epimorphism. It follows that the composition

$$\ker\delta \xrightarrow{(2.1.5)} K' \longrightarrow D'$$

is zero (for, if $\varphi\circ\psi = 0$ with $\psi$ and epimorphism, then $\varphi = 0$, and this is seen by "cancelling" the epimorphism $\psi$ from both sides of the equation $\varphi\circ\psi = 0\circ\psi$). Now according to (2.1.4), $\operatorname{coker}\beta = \ker(K' \to D')$, and hence we have a unique map

$$\partial\colon \ker\delta \to \operatorname{coker}\beta$$

such that (2.1.5) is the composite $\ker\delta \xrightarrow{\partial} \operatorname{coker}\beta \hookrightarrow K'$. Thus the connecting homomorphism satisfying (1), (2), and (3) of the statement is shown to exist. It remains to prove that it fits into the six term exact sequence given in the statement of the Snake Lemma. We claim it is enough to prove the exactness of the three term complex

$$(\dagger) \qquad \ker\gamma \longrightarrow \ker\delta \xrightarrow{\partial} \operatorname{coker}\beta.$$

Indeed, if the above complex is exact, then by duality, the complex

$$\ker\delta \xrightarrow{\partial} \operatorname{coker}\beta \to \operatorname{coker}\gamma$$

is also exact. Using (2.1.1) and (2.1.2), we are done. To show that (†) is exact, it is enough to show that

$$(\ddagger) \qquad \ker\gamma \longrightarrow \ker\delta \xrightarrow{(2.1.5)} K'$$

is exact since $\operatorname{coker}\beta \to K'$ is a monomorphism by (2.1.4).

We now show that (‡) is exact. First note that $\ker\gamma = \ker(K \to C')$, where $K \to C'$ is the obvious map, i.e., the composite $K \hookrightarrow C \xrightarrow{\gamma} C'$. There are many ways to see this. For example, consider the commutative exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\gamma"] & D' \arrow[d] \\
0 \arrow[r] & C' \arrow[r, equal] & C' \arrow[r] & 0
\end{tikzcd}$$

By Problem (2) (b) of HW-5 we get an exact sequence

$$0 \to \ker(K \to C') \longrightarrow \ker\gamma \longrightarrow D'.$$

The last arrow is clearly zero, since it factors as $\ker\gamma \to C \xrightarrow{\gamma} C' \to D'$. Hence $\ker\gamma = \ker(K \to C')$. Next consider the exact commutative diagram:

$$\begin{tikzcd}
B \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & \ker\delta \arrow[r] \arrow[d, "(2.1.5)"] & 0 \arrow[d, equal] \\
B \arrow[r] \arrow[d] & C' \arrow[r] & K' \arrow[r] & 0 \\
0 & & &
\end{tikzcd}$$

where the unlabelled arrows are the obvious ones. By Problem 4 of HW-5, and using the fact that $\ker(K \to C') = \ker\gamma$, we get an exact sequence

$$\ker\gamma \longrightarrow \ker(\ker\delta \to K') \longrightarrow 0.$$

It follows that (‡) is exact as required. Indeed we proved in class that if $P \to Q$ and $Q \to R$ are maps in $\mathscr{C}$, then to say $P \to \ker(Q \to R) \to 0$ is exact is equivalent to saying $P \to Q \to R$ is exact.