SOME BASIC RESULTS FOR EXACT CATEGORIES

Throughout $\mathscr{C}$ is an exact category.

(1) Let $A \in \mathscr{C}$. We have

  (a) $\operatorname{coker}(0 \to A) = (A \xrightarrow{\mathbf{1}_A} A)$.

  (b) $\ker(A \to 0) = (A \xrightarrow{\mathbf{1}_A} A)$.

Proof. By duality, it is enough to prove (a). Let $f\colon A \to T$ be any map in $\mathscr{C}$ (note that the composite $0 \to A \xrightarrow{f} T$ is necessarily zero). It is clear that f factors uniquely through $\mathbf{1}_A$. □

(2) Let $f\colon A \to B$ be a map in $\mathscr{C}$.

  (a) The following are equivalent:

    (i) The sequence $0 \to A \xrightarrow{f} B$ is exact.

    (ii) The map f is a monomorphism.

    (iii) $\ker(f) = 0$

  (b) The following are equivalent:

    (i) The sequence $A \xrightarrow{f} B \to 0$ is exact.

    (ii) The map f is an epimorphism.

    (iii) $\operatorname{coker}(f) = 0$.

Proof. Clearly (b) follows from (a) by duality. For (a), suppose (i) is true. Consider the standard factorisation of f:

$$A \twoheadrightarrow \mathbf{coim}(f) \xrightarrow{\sim} \mathbf{im}(f) \hookrightarrow B.$$

Since the composites of monomorphisms is a monomorphism (using the cancellation from left of monomorphisms), we only have to show that $A \to \mathbf{coim}(f)$ is a monomorphism. But by part (a) of Problem 1, this is the map $\mathbf{1}_A\colon A \to A$. Hence f is a monomorphism giving (ii).

Next suppose (ii) is true, i.e., f is a monomorphism. Then $0 \to A$ satisfies the universal property for $\ker(f)$. Indeed, if $g\colon T \to A$ is a map such that $f \circ g = 0$, then $f \circ g = f \circ 0$ and we can cancel the f from the left by definition of monomorphism. Thus $g = 0$ and hence factors as $T \to 0 \to A$. Thus (iii) is true.

We make the following general observation now. Let $D \in \mathscr{C}$ be any object. Note that $\mathbf{1}_D\colon D \to D$ is a monomorphism. If we apply (ii) $\Rightarrow$ (iii) to the special case of $f = \mathbf{1}_D$, we see that $\ker(\mathbf{1}_D) = 0$. By Problem 1(a) it follows that

$$\mathbf{im}(0 \to D) = 0. \tag{$*$}$$

Now suppose (iii) is true, i.e., $\ker(f) = 0$. In view of the above equality $0 \to A \xrightarrow{f} B$ is exact. □

(3) Let $A \in \mathscr{C}$.

  (a) $\mathbf{im}(0 \to A) = 0$.

(b) $\mathbf{coim}(A \to 0) = 0$.

Proof. Part (a) is simply $(*)$. Part (b) is the dual of part (a). $\square$

(4) Let $f\colon A \to B$ be a map in $\mathscr{C}$. Then
(a) $0 \to \ker(f) \to A \xrightarrow{f} B$ is exact.
(b) $A \xrightarrow{f} B \to \operatorname{coker}(f) \to 0$ is exact.

Proof. As usual, we only prove (a), and note that this is sufficient. We make the following general observation. Suppose $0 \to X \to Y$ is exact. Then by Problem 2, $\ker(X \to Y) = 0$ and hence by Problem 1(a), $\mathbf{im}(X \to Y) = X$. In our case, since $\ker(f) \to A$ is a monomorphism by definition of kernels, Problem 2(a) gives us that $0 \to \ker(f) \to A$ is exact. From what we just said, $\mathbf{im}(\ker(f) \to A) = \ker(f)$. It follows that $0 \to \ker(f) \to A \xrightarrow{f} B$ is exact. $\square$

(5) Let $f\colon A \to B$ be a map in $\mathscr{C}$. Let $p\colon A \twoheadrightarrow \mathbf{coim}(f)$ be the natural epimorphism and $i\colon \mathbf{im}(f) \hookrightarrow B$ the natural monomorphism.
(a) If f is a monomorphism then $p\colon A \to \mathbf{coim}(f)$ is an isomorphism.
(b) If f is an epimorphism then $i\colon \mathbf{im}(f)$ is an isomorphism.
(c) If f is a monomorphism and an epimorphism it is an isomorphism.

Proof. It is enough to prove (a) for (b) follows by duality and (c) by (a), (b), the standard factorisation of f, and the fact that in an exact category the natural map $\mathbf{coim}(f) \to \mathbf{im}(f)$ is an isomorphism.

To prove (a), suppose f is a monomorphism. Then $\ker(f) = 0$ by Problem 2(a). By Problem 1(a), we get $\mathbf{coim}(f) = \operatorname{coker}(0 \to A) = A$. $\square$

(6) Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be two maps in $\mathscr{C}$.
(a) If $\beta \circ \alpha\colon A \to C$ is a monomorphism, then so is $\alpha\colon A \to B$.
(b) If $\beta \circ \alpha\colon A \to C$ is an epimorphism, then so is $\beta\colon B \to C$.

Proof. As usual it is enough to prove (a). Suppose $f, g\colon T \rightrightarrows A$ are two maps such that $\alpha \circ f = \alpha \circ g$. Then $\beta \circ \alpha \circ f = \beta \circ \alpha \circ g$. Since $\beta \circ \alpha$ is a monomorphism, we get $f = g$. $\square$

(7) Suppose $i\colon S \to A$ is a monomorphism, $\alpha\colon A \to B$ a map. Suppose further that $\ker(\alpha) = (K, j)$ and j factors as a composite $K \xrightarrow{\varphi} S \xrightarrow{i} A$. Then $(K, \varphi) = \ker(\alpha \circ i)$.

em Proof. Suppose $g\colon T \to S$ is a map such that $(\alpha \circ i) \circ g = 0$. Then $\alpha \circ (i \circ g) = 0$ and since $(K, i) = \ker(\alpha)$ we have a unique map $\gamma\colon T \to K$ such that $j \circ \gamma = i \circ g$. Since $j = i \circ \varphi$, we get $i \circ \varphi \circ \gamma = i \circ g$. Since i is a monomorphism, we get $\varphi \circ \gamma = g$. If $\gamma'\colon T \to K$ is another map such that $\varphi \circ \gamma' = g$, then the above steps can be reversed to get $j \circ \gamma' = i \circ g = j \circ \gamma$, which means $\gamma = \gamma'$. $\square$

(8) Let

$$A \longrightarrow B \longrightarrow C$$

be a complex in $\mathscr{C}$. Then the following are equivalent.
(a) $A \longrightarrow B \longrightarrow C$ is exact.
(b) $A \longrightarrow \ker(B \to C) \longrightarrow 0$ is exact.

(c) $0 \longrightarrow \operatorname{coker}(A \to B) \longrightarrow C$ is exact.

Proof. By duality it is enough to prove that (a) and (b) are equivalent. Let us write f for the map $A \to B$ and set $K = \ker(B \to C)$. We have the standard factorisation of $f\colon A \to B$ as

$$A \twoheadrightarrow \mathbf{coim}(f) \xrightarrow{\sim} \mathbf{im}(f) \hookrightarrow B$$

with the first arrow an epimorphism and the last arrow a monomorphism. In particular, since the middle arrow is an isomorphism, $A \to \mathbf{im}(f)$ is an epimorphism. Since the map $A \to C$ is zero, and $A \to \mathbf{im}(f)$ is an epimorphism, it follows that $\mathbf{im}(f) = 0$. Indeed if $\pi\colon A \to \mathbf{im}(f)$ is the epimorphism we are talking about and $h\colon \mathbf{im}(f) \to C$ the composite $\mathbf{im}(f) \to B \to C$, then $h \circ \pi = 0 = 0 \circ \pi$, and π can be cancelled from the right, being an epimorhism, to give $h = 0$ as asserted. We therefore have a map $\mathbf{im}(f) \to K$, which is necessarily a monomorphism by Problem 6(a). We thus have a factorisation of f as follows:

$$A \twoheadrightarrow \mathbf{coim}(f) \xrightarrow{\sim} \mathbf{im}(f) \hookrightarrow K \hookrightarrow B$$

Now suppose (a) is true, i.e., $A \longrightarrow B \longrightarrow C$ is exact. Then $K = \mathbf{im}(f)$ and hence the natural map $A \to K$ is an epimorphism. By (ii) $\Rightarrow$ (i) of Problem 2(b), we get that $A \to K \to 0$ is exact, and hence (b) is true.

Conversely, suppose $A \to K \to 0$ is exact. Then $A \to K$ is an epimorphism. On the other hand $A \to K$ factors as $A \to \mathbf{im}(f) \hookrightarrow K$. By Problem 6 (b) we get $\mathbf{im}(f) \to K$ is an epimorphism. Thus $\mathbf{im}(f) \to K$ is a monomorphism and an epimorphism. By Problem 5(c) it is an isomorphism, i.e., $\mathbf{im}(f) = K$. This means $A \to B \to C$ is exact. □

(9) Suppose $g\colon B \to C$ is a map in $\mathscr{C}$.
(a) If $j\colon C \to D$ is a monomorphism, then $\ker(j \circ g) = \ker(g)$.
(b) If $p\colon A \to B$ is an epimorphism, then $\operatorname{coker} g \circ p) = \operatorname{coker}(g)$.

Proof. As usual, it is enough to prove (a). Let $(K, i) = \ker(g)$. Let $h\colon T \to B$ be a map such that $(j \circ g) \circ h = 0$. This means $j \circ (g \circ h) = 0 = i \circ 0$. Since j is a monomorphism, this yields $g \circ h = 0$. Since $(K, i) = \ker(g)$, there is a unique map $s\colon T \to K$ such that $i \circ s = h$. This proves (a). □