EXACT, ADDITIVE, AND ABELIAN CATEGORIES

For some of you, this is your first exposure to abstract nonsense related to homological algebra, and derived functors. We are interested in defining kernels and cokernels (as well as images etc) in settings which are more general than sets, and yet "occur in nature". Here are some basic definitons. Some of the Homework problems are based on these definitions. The important thing to keep in mind is that our objects need not be sets (a sheaf is a-priori not a set), and yet one wishes to define kernels, cokernels, etc.

Kernels, cokernels, images, and coimages

Zero Objects. Let $\mathscr{C}$ be a category. An *initial object* in $\mathscr{C}$ is an object Z such that if A is any object in $\mathscr{C}$, there is precisely one morphism $Z \to A$. An example is $Z = \mathbb{Z}$, the ring of integers, in the category $\mathscr{C} = \mathbf{Ring}$—the category of rings with unity. A *final object* in $\mathscr{C}$ is an object $\bigstar$ such that for every object A, there is precisely one morphism $A \to \bigstar$ in $\mathscr{C}$. The usual example is the category of sets, with $\bigstar$ being any set with exactly one element. For definiteness pick $\bigstar = \{*\}$, if you wish.

An object which is both an initial object and a final object in $\mathscr{C}$ is called a *zero object* of $\mathscr{C}$. Suppose $\mathscr{C}$ has a zero object 0. Suppose A and B are two objects, then the *zero morphism* $A \to B$ is one that can be factored as $A \to 0 \to B$.

Assumption. For the remaining part of this section the category $\mathscr{C}$ will be assumed to have a zero object which we will denote 0. Zero morphisms will also be denoted by the same symbol, viz., 0. "Map" is shorthand for "morphism".

Kernels and Cokernels. Let $f\colon A \to B$ be a map (i.e. a morphism) in $\mathscr{C}$. The *kernel of* f—denoted $\ker f$—is a pair (K, i), with K an object of $\mathscr{C}$, $i\colon K \to A$ a monomorphism[1] such that $f \circ i = 0$ and whenever $g\colon T \to A$ is a map in $\mathscr{C}$ with $f \circ g = 0$, there is a map $g'\colon T \to K$ such that $g = i \circ g'$. Note that since i is a monomorphism, g' is unique. Thus kernels have a universal property. It follows that if (K, i) and (K', i') are kernels for f, then there is *unique isomorphism* $\theta\colon K \xrightarrow{\sim} K'$ such that $i' \circ \theta = i$ (draw a picture). This partially justifies the use of the definite article "the" in the phrase "the kernel of f".

For f as above, the *cokernel of* f, $\mathrm{coker} f$, is a pair (C, p), with $p\colon B \to C$ an epimorphism[2] such that $p \circ f = 0$ and whenever $g\colon B \to T$ is a map such that $g \circ f = 0$, then there is a map $g'\colon C \to T$ such that $g = g' \circ p$. As in an earlier argument, g' is necessarily unique, and (C, p) (as a pair) is unique up to unique isomorphism. And as before, the use of the definite article "the" is only partially justified[3].

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[1] i.e., $i \circ g = i \circ g' \implies g = g'$.

[2] i.e., $g \circ p = g' \circ p \implies g = g'$.

[3] This is one of the little grammatical infelicities we will just have to get used to–we'll use the definite article "the" all over the place, in a somewhat inaccurate way. That is the price of going "categorical".

It is not hard to see that the above concepts agree with our usual notions of kernels and cokernels in the category of groups, vector spaces, and modules over a ring.

Images and Coimages. Assume all maps in $\mathscr{C}$ have kernels and cokernels. The *image* of a map $f\colon A \to B$ in $\mathscr{C}$ is the kernel of the cokernel (or, more accurately, a kernel of a cokernel). The *coimage* of f is a cokernel of a kernel. The image of f is denoted $\mathbf{im} f$ and coimage of f is denoted $\mathbf{coim} f$.

Canonical factorization. Every map $f\colon A \to B$ in $\mathscr{C}$ (which we assume has kernels and cokernels for every map) has a *canonical factorization*

$$A \to \mathbf{coim} f \xrightarrow{f'} \mathbf{im} f \to B.$$

Various kinds of categories

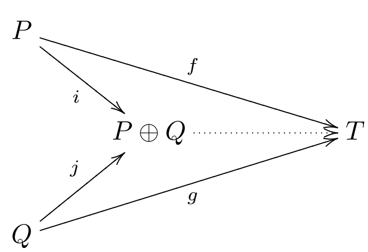
Our aim is to do homological algebra in a setting that is not too specific, and on the other hand not so general as to lose all structure. It turns out that sheaves of abelian groups form an *abelian category* which is the right framework for much of homological algebra. In what follows, we lead up to abelian categories in small steps.

For any pair of objects A, B in a category $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ will denote the class of morphisms from A to B in $\mathscr{C}$.

Exact categories. A category $\mathscr{C}$ is said to be an *exact category* if is has a zero object, every map in $\mathscr{C}$ has a kernel as well as a cokernel, and for $f\colon A \to B$ in $\mathscr{C}$, the canonical map $f'\colon \mathbf{coim} f \to \mathbf{im} f$ is an isomorphism.

Additive categories. A category $\mathscr{C}$ is said to be *additive* if the following conditions are satisfied.

- $\mathscr{C}$ has a zero object.
- $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is an additive group for every pair of objects A and B in $\mathscr{C}$, with the zero morphism $A \xrightarrow{0} B$ being the zero element of $\mathrm{Hom}_{\mathscr{C}}(A, B)$.
- If f and g are in $\mathrm{Hom}_{\mathscr{C}}(A, B)$, then for every $\varphi\colon T \to A$ and every $\psi\colon B \to T'$, we have
$$(f + g) \circ \varphi = f \circ \varphi + g \circ \varphi \qquad \psi \circ (f + g) = \psi \circ f + \psi \circ g.$$
- For any two objects P and Q in $\mathscr{C}$, the *direct sum* of P and Q, $P \oplus Q$ exists in $\mathscr{C}$, i.e., there are maps $i\colon P \to P \oplus Q$ and $j\colon P \to P \oplus Q$, such that if $f\colon P \to T$ and $g\colon Q \to T$ are two maps, then there is a unique map $f \oplus g\colon P \oplus Q \to T$ such that $(f \oplus g) \circ i = f$ and $(f \oplus g) \circ j = g$. In other words, there exists a unique way of filling the dotted arrow to make the diagram below commute (and the unique solution is denoted $f \oplus g$).



Abelian categories. A category which is additive and exact is called an *abelian category.*