NOTES 2

Throughout A is a ring. As usual "maps" mean A-module homomorphisms.

Direct Systems and Direct Limits

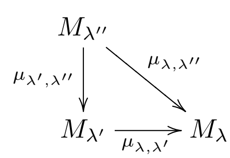
Definitions. A partially ordered set $(\Lambda, \prec)$ is said to be a *directed set* if given a pair of elements λ_1 and λ_2 in Λ, there exists $\lambda \in \Lambda$ such that $\lambda_i \prec \lambda$ for $i = 1, 2$. Given a directed set $(\Lambda, \prec)$, a *direct system* of A-modules is a family of modules $(M_\lambda)_{\lambda\in\Lambda}$ together with A-maps $\mu_{\lambda,\lambda'} \colon M_{\lambda'} \to M_\lambda$, one for every pair of elements $(\lambda', \lambda) \in \Lambda \times \Lambda$ with $\lambda' \prec \lambda$, such that $\mu_{\lambda,\lambda} = \mathbf{1}_{M_\lambda}$, and if $\lambda'' \prec \lambda' \prec \lambda$ then $\mu_{\lambda,\lambda'} \circ \mu_{\lambda',\lambda''} = \mu_{\lambda,\lambda''}$.

If $(M_\lambda, \mu_{\lambda,\lambda'})$ is direct system, then the *direct limit* of this direct system is a module M, together with maps $\mu_\lambda \colon M_\lambda \to M$, one for each $\lambda \in \Lambda$, satisfying $\mu_\lambda \circ \mu_{\lambda,\lambda'} = \mu_{\lambda'}$ for $\lambda' \prec \lambda$ and such that if we have a family of A-maps $\nu_\lambda \colon M_\lambda \to N$, $\lambda \in \Lambda$, where N is an A-module, then there exists a unique A-map $\nu \colon M \to N$ such that $\nu \circ \mu_\lambda = \nu_\lambda$, $\lambda \in \Lambda$. If M^* is another A-module with this property, i.e., if we have A-maps $\mu^*_\lambda \colon M_\lambda M^*$, $\lambda \in \Lambda$ satisfying $\mu^*_\lambda \circ \mu_{\lambda,\lambda'} = \mu^*_{\lambda'}$, $\lambda' \prec \lambda$, and such that if (N, ν_λ) is as above, there is a unique map $\nu^* \colon M^* \to N$ satisfying $\nu^* \circ \mu^*_\lambda = \nu_\lambda$, $\lambda \in \Lambda$, then clearly we have a unique isomorphism $\varphi \colon M \xrightarrow{\sim} M^*$ such that $\mu^*_\lambda = \varphi \circ \mu_\lambda$, $\lambda \in \Lambda$. To see this, first note that by the universal property of M we have a unique map $\varphi \colon M \to M^*$ satisfying $\mu^*_\lambda = \varphi \circ \mu_\lambda$, $\lambda \in \Lambda$. Similarly we have a unique map $\psi \colon M^* \to M$ satisfying $\mu_\lambda = \psi \circ \mu^*_\lambda$, $\lambda \in \Lambda$. Then $\theta = \psi \circ \varphi \colon M \to M$ satisfies $\mu_\lambda = \theta \circ \mu_\lambda$, $\lambda \in \Lambda$. By the uniqueness of such a θ, it follows that $\theta = \mathbf{1}_M$. Similarly $\varphi \circ \psi = \mathbf{1}_{M^*}$. Hence direct limits are *unique up to unique isomorphisms.*

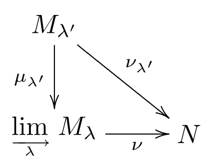
If M is the direct limit of (M_λ), we write

$$M = \varinjlim_{\lambda} M_\lambda.$$

Schematically, a direct system $(M_\lambda, \mu_{\lambda,\lambda'})$ is represented by commutative diagrams (one for every triple of indices $\lambda'' \prec \lambda' \prec \lambda$)



Similarly, if $(\nu_\lambda \colon M_\lambda \to N)$ is a family of maps satisfying $\nu_\lambda \mu_{\lambda,\lambda'} = \nu_{\lambda'}$, then for each λ' the following diagram commutes



and ν is the only map with this property.

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Cofinal systems. Let $(\Lambda, \prec)$ be a directed set. A sub-directed set $\Gamma \subset \Lambda$ is said to be *cofinal* with respect to Λ, if given $\lambda \in \Lambda$ there exists a $\gamma \in \Gamma$ with $\lambda \prec \gamma$. In what follows we fix a directed set Λ and a cofinal subset Γ of Λ.

Maps of direct systems. Let $(\Lambda, \prec)$ be a directed set. If $(M_\lambda)_{\lambda\in\Lambda}$ and $(N_\lambda)_{\lambda\in\Lambda}$ are direct systems of A-modules, then $\mathrm{Hom}_\Lambda((M_\lambda),(N_\lambda))$ is the group of maps between direct systems, i.e., an element of the above set is a collection of maps $(\varphi_\lambda\colon M_\lambda \to N_\lambda)$ compatible with the direct system structures on (M_λ) and (N_λ). Every module T will be regarded as a direct system, namely as the constant direct system. Thus the symbol $\mathrm{Hom}_\Lambda((M_\lambda),T)$ makes sense. In particular, by definition of a direct limit

$$\mathrm{Hom}_\Lambda((M_\lambda),\, T) \xrightarrow{\sim} \mathrm{Hom}_A(\varinjlim_{\lambda\in\Lambda} M_\lambda,\, T)$$

for every A-module T.

Existence. Suppose (M_λ) is a direct system of A-modules. There are two constr-cuctions (at least) of $\varinjlim_\lambda M_\lambda$. In the first construction, consider the set

$$X = \coprod_{\lambda\in\Lambda} M_\lambda$$

which is the *disjoint union* of the M_λ. Note that given $x \in X$, it lies in a unique M_λ and let us denote the index λ where this occurs by $\lambda(x)$. Next define an equivalence relation "$\sim$" on X as follows. If x and y are elements of X, we say $x \sim y$ if there exists an index λ such that $\lambda(x) \prec \lambda$ and $\lambda(y) \prec \lambda$ and $\mu_{\lambda,\lambda(x)}(x) = \mu_{\lambda,\lambda(y)}(y)$. One checks that this gives an equivalence relation on X. Set

$$\varinjlim_\lambda M_\lambda = X/\!\sim .$$

One checks that $X/\!\sim$ has the structure of an A-module given by $[x] + [y] = [\mu_{\lambda,\lambda(x)}(x) + \mu_{\lambda,\lambda(y)}(y)]$ where λ is an index greater than or equal both $\lambda(x)$ as well as $\lambda(y)$ ($[z]$ denoting the equivalence class of $z \in X$), for $[x],[y] \in X/\!\sim$, and $a[x] = [ax]$ for $[x] \in X/\!\sim$.

The second construction is as follows. Let

$$\widetilde{M} = \bigoplus_{\lambda\in\Lambda} M_\lambda.$$

For $x \in \widetilde{M}$, let the *support* of x be the set

$$\mathrm{Supp}\,(x) = \{\lambda \in \Lambda \mid \pi_\lambda(x) \neq 0\}$$

where $\pi_\lambda\colon \widetilde{M} \to M_\lambda$ is the projection map. Let $\lambda^* \in \Lambda$. We say $\mathrm{Supp}\,(x) \prec \lambda^*$ if $\lambda \prec \lambda^*$ for every $\lambda \in \mathrm{Supp}\,(x)$. Since $\mathrm{Supp}\,(x)$ is finite (by definition of a direct sum), such a λ^* always exists. Let K be the submodule of $\widetilde{M}$ given by

$$K = \{x \in \widetilde{M} \mid \exists\ \lambda^* \succ \mathrm{Supp}\,(x) \text{ such that } \sum_{\lambda\in\mathrm{Supp}\,(x)} \mu_{\lambda^*,\lambda}\pi_\lambda(x) = 0\}.$$

Set

$$\varinjlim_\lambda M_\lambda = \widetilde{M}/K.$$

Localisation

Let A be a commutative rung, M an A-module, and $S \subset A$ a subset which is multiplicatively closed. This means $1 \in S$ and $st \in S$ whenever s and t are in S. The module *localisation of M at S* is the set of equivalence classes of pairs $(x, s) \in M \times S$ where (x, s) and (y, t) are related if there exists $a \in S$ such that $a(tx - sy) = 0$. Once writes x/s or $\frac{x}{s}$ for the equivalence class of (x, s). S^1M can be made into a module by defining addition and scalar multiplication by the rules:

$$\frac{x}{s} + \frac{y}{t} = \frac{tx + sy}{st}$$

and

$$a(\frac{x}{s}) = \frac{ax}{s}.$$

One checks that these operations are well defined. Since A is itself an A-module, S^1A makes sense. It is easy to see that $S^{-1}A$ is a commutative ring under the obvious product (namely, $(a/s)(b/t) = (ab)/(st)$) and one checks easily that $S^{-1}M$ is an $S^{-1}A$-module.

If $t \in A$ and $S = \{1, t, t^2, \dots\}$, then it is traditional to write M_t instead of $S^{-1}M$.

Let $q\colon A \to S^{-1}S$ be the "localisation map" $a \mapsto a/1$. Then q is a ring homomorphism. The localisation map generalises to the module M and gives us an A-module map $q_M\colon M \to S^{-1}M$ given by $m \mapsto m/1$.

The localisation $S^{-1}A$ has the following universal property. If $f\colon A \to B$ is a ring homomorphism (between commutative rings) and $f(s)$ is a unit in B for every $s \in S$, then there exists a unique map $f'\colon S^1A \to B$ such that $f' \circ q = f$. In fact the map f' is $a/s \mapsto f(s)^{-1}f(a)$. Similarly, if $\varphi\colon M \to N$ is an A-module map such that the maps $\mu_s\colon \varphi(N) \to \varphi(N)$, $s \in S$, given by $\mu_s(x) = sx$, are all (bijective) isomorphisms, then there exists a unique map $\varphi'\colon S^{-1}M \to N$ such that $\varphi = \varphi' \circ q_M$.

It is not hard to see that $S^{-1}M \xrightarrow{\sim} (S^{-1}A) \otimes_A M$. We usually write $S^{-1}M = (S^{-1}A) \otimes_A M$.

If $t \in A$, then one sees from the universal properties of localisation that $A_t = A[X]/(1 - tX)$, and similarly $M_t = M[X]/(1 - tX)M[X]$, where $M[X] := A[X] \otimes_A M$.