INJECTIVES AND DERIVED CATEGORIES

Summary of things done

Main Theorems. Let $\mathscr{A}$ be an abelian category and $\mathbf{C}(\mathscr{A})$ the category of complexes of objects in $\mathscr{A}$. We regard every object A of $\mathscr{A}$ as a complex with A in the 0^{th} spot and 0 in all other spots of the complex. It is clear that a map $A \to A'$ in $\mathscr{A}$ can be regarded as a map in $\mathbf{C}(\mathscr{A})$. Thus we have a functor $\imath\colon \mathscr{A} \hookrightarrow \mathbf{C}(\mathscr{A})$ which is an embedding.

The main results we proved were:

Theorem 1. *Suppose $\mathscr{A}$ is an abelian category with enough injectives, A, B objects in $\mathscr{A}$, $A \xrightarrow{c} C^\bullet$ a resolution of A and $B \xrightarrow{b} I^\bullet$ a map of complexes and $\psi\colon A \to B$ a map in $\mathscr{A}$. Then there is a map $\varphi\colon C^\bullet \to I^\bullet$, unique up to homotopy, which lifts ψ, i.e., there is a homotopy unique map $\varphi\colon C^\bullet \to I^\bullet$ such that the diagram of complexes*

$$\begin{array}{ccc} A & \xrightarrow{c} & C^\bullet \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[b]{} & I^\bullet \end{array}$$

commutes.

Theorem 2 (The Horseshoe Lemma)**.** *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence in an abelian category $\mathscr{A}$ with enough injectives. If $\alpha\colon A \to I^\bullet_A$ and $\gamma\colon C \to I^\bullet_C$ are injective resolutions, then we can find an injective resolution $\beta\colon B \to J^\bullet$ such that we have a commutative diagram in $\mathbf{C}(A)$ with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^\bullet_A & \longrightarrow & J^\bullet & \longrightarrow & I^\bullet_C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Injective resolution functor. Let $\mathscr{A}$ be an abelian category. Define $\mathbf{K}(\mathscr{A})$ to be the category whose objects are complexes of objects from $\mathscr{A}$, and whose morphisms are *homotopy equivalance classes* of maps in $\mathbf{C}(A)$. The embedding $\imath\colon \mathscr{A} \to \mathbf{C}(\mathscr{A})$ mentioned above, clearly descends to $\mathbf{K}(\mathscr{A})$ and we denote this functor also by $\imath$, and it too is an embedding. Thus we have

$$\mathscr{A} \hookrightarrow \mathbf{K}(\mathscr{A}).$$

Suppose $\mathscr{A}$ has enough injectives. If one picks an injective resolution $\varphi(A)\colon A \to \lambda(A)$ for each $A \in \mathscr{A}$ ($\lambda(A) \in \mathbf{K}(\mathscr{A})$, then according to Theorem 1, a map $A \to A'$ lifts uniquely to a map $\lambda(A) \to \lambda(A')$ in $\mathbf{K}(\mathscr{A})$, and again by Theorem 1, a composite $A \to A' \to A''$ gives rise to a corresponding composite $\lambda(A) \to \lambda(A') \to \lambda(A'')$ in $\mathbf{K}(\mathscr{A})$. Thus we have:

- A functor $\lambda\colon \mathscr{A} \to \mathbf{K}(\mathscr{A})$ taking values in injective complexes (in fact bounded below injective complexes), and
- a map of functors
$$\varphi\colon \imath \to \lambda$$
such that for each $A \in \mathscr{A}$, $\varphi(A)$ is an injective resolution.

Definition 1. A pair (λ, φ) is said to be an *injective resolution functor* it has the above two properties.

Theorem 1 (yet again!) gives us (for an abelian category with enough injectives):

Theorem 3. *Two injective resolution functors are unique up to unique isomorphism. In greater detail, if* (λ, φ) *and* $(\lambda' \varphi')$ *are two injective resolution functors, then there is a unique isomorphism* $\psi\colon \lambda \xrightarrow{\sim} \lambda'$ *such that the diagram*

$$\begin{array}{ccc} \imath & = & \imath \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ \lambda & \underset{\psi}{\xrightarrow{\sim}} & \lambda' \end{array}$$

commutes.

Derived functors. Now suppose $\mathscr{A}$ and $\mathscr{B}$ are abelian categories, $\mathscr{A}$ with enough injectives, and $T\colon \mathscr{A} \to \mathscr{B}$ an *additive functor*. Since T is additive, it respects homotopies and hence we have a functor (also denoted T) $T\colon \mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$ given by $C^\bullet \mapsto T(C^\bullet)$. Pick an injective resolution functor (λ, φ). For each $i \in \mathbb{Z}$ define the i^{th} *right derived functor of* T to be the additive functor $R^iT\colon \mathscr{A} \to \mathscr{B}$ given by the formula
$$R^iT := \mathrm{H}^i(T \circ \lambda).$$

It is immediate from Theorem 3 that R^iT is independent (up to unique isomorphism) of the chosen injective resolution functor. We will not belabour this point, and will from now on take a naive attitude towards computing derived functors. The Horsehoe Lemma then gives

Theorem 4. *Given a short exact sequence*
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
in $\mathscr{A}$*, we have a long exact sequence*
$$\cdots \to R^{i-1}T(C) \xrightarrow{\delta} R^iT(A) \xrightarrow{R^iT(f)} R^iT(B) \xrightarrow{R^iT(g)} R^iT(C) \xrightarrow{\delta} R^{i+1}T(A) \to \ldots$$
and the "connecting maps" δ *are natural transformations.*

The last statement (on the naturality of δ) is easy to prove, once one examines the behaviour of the Horshoe Lemma when one varies the short exact sequence in $\mathscr{A}$. I made the right statement in class.

More on Derived Functors

Let $T\colon \mathscr{A} \to \mathscr{B}$ be an additive functor between abelian categories with $\mathscr{A}$ having enough injectives.

1) If T is left exact then $T^0 = T$.

2) If A is an injective object, then $R^iTA = 0$ for $i \geq 1$. Indeed we can pick $\lambda(A)$ to be A, i.e., we can work with the injective resolution $1_A \colon A \to A$ of A. Then applying T and taking cohomology, the assertion follows.

3) A collection of additive functors $\{S^i\}_{i\geq 0}$ from $\mathscr{A} \to \mathscr{B}$ is said to be a δ-functor if it transforms a short exact sequence as in the previous Theorem to a long exact sequence

$$\cdots \to S^{i-1(C)} \xrightarrow{\delta} S^i(A) \xrightarrow{S^i(f)} S^i(B) \xrightarrow{S^i(g)} S^i(C) \xrightarrow{\delta} S^{i+1}(A) \to \ldots$$

in a functorial manner, i.e. the above assignment should be a functor from the category of short exact sequences in A to the category of long exact sequences in B. We don't need A to have enough injectives to make this definition. The Theorem then says that $\{R^iT\}$ is a δ-functor. Note that if $\{S^i\}$ is a δ-functor, then S^0 is necessarily left exact.

4) In fact $\{R^iT\}$ is a *universal δ-functor*. This means that given a δ-functor $\{S^i\}$ and a natural transformation $T \to S^0$, there are unique maps $R^iT \to S^i$ $i \geq 0$ giving a map of δ-functors. Indeed, given an object $A \in \mathscr{A}$, pick an embedding $A \hookrightarrow I$ of A into an injective object I. Let $C = \operatorname{coker}(A \to I)$. Now consider the commutative diagram of solid arrows (arising from the natural transformation $T \to S^0$) and exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & TA & \longrightarrow & TI & \longrightarrow & TC & \longrightarrow & R^1TA & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \vdots & & \\
0 & \longrightarrow & S^0A & \longrightarrow & S^0I & \longrightarrow & S^0C & \longrightarrow & S^1A & \longrightarrow &
\end{array}$$

The top row is exact on the right because $R^1TI = 0$, I being injective. Thus $R^1TA = \operatorname{coker}(TI \to TC)$. By definition of a cokernal, we have a unique map which fills the dotted arrow to make the whole diagram commute. I leave it to you to show (a) the map $R^1TA \to S^1A$ so defined is independent of I; (b) that it actually defines a functorial map; and (c) if $0 \to A \to B \to C \to 0$ is an exact sequence then the required diagram between the two δ-functors commutes up to *level* 1.

Now proceed by induction to finish the proof—if $R^i \to S^i$ has been defined for $i \leq k$ in such a way that for a short exact sequence of objects in $\mathscr{A}$, we have a map *up to levek* k of the corresponding long exact sequences of the two δ-functors as required. Since $R^iTI = 0$ for $i \geq 1$, we have an isomorphism $R^kTC \xrightarrow{\sim} R^{k+1}TA$. One then defines $R^{k+1}TA \to S^{k+1}A$ as the unique map which makes the diagram below commute:

$$\begin{array}{ccc}
R^kTC & \xrightarrow{\sim} & R^{k+1}TA \\
\downarrow & & \downarrow \\
S^kC & \longrightarrow & S^{k+1}A
\end{array}$$

Once again, one checks that this map $R^{k+1}A \to S^{k+1}A$ is independent of I and defines a natural transformation $R^{k+1} \to S^{k+1}$ with the required properties.

5) We can dualize and talk about *left derived functors* L_i of additive functors when $\mathscr{A}$ has enough projectives. In greater detail, if A has enough projectives (it may or may not have enough injectives), we can get a projective resolution functor (τ, φ) with $\tau \colon \mathscr{A} \to \mathbf{K}(\mathscr{A})$ a functor and $\varphi \colon \tau \to \imath$ a natural transformation

such that for each $A \in \mathscr{A}$ $\varphi(A)\colon \tau(A) \to \imath(A) = A$ is a projective resolution. If $T\colon \mathscr{A} \to \mathscr{B}$ is additive we can define

$$L_i T = \mathrm{H}^{-i}(T \circ \tau).$$

Again, a suitable definition of δ-functors can be made, which are negatively indexed or else, have their indices as subscripts rather than superscripts as in $\{L_i T\}$ and these are also universal δ-functors. I leave it to you to formulate the universal property. If $\{S_i\}_{i \geq 0}$ is a "homology" δ-functor, your definition should imply that S_0 is necessarily right exact.

6) There is obvious way of talking about right derived functors of contravariant additive functors using projective resolutions, if the category has enough projectives. Similarly, one can talk about left derived functors of contravariant functors using injectives. I leave these details to you. Again, if T is left exact and contravariant, then $R^0 T = T$, and if it is right exact, $L_0 = T$.

Examples of derived functors

The bifunctor $\mathrm{Tor}_i^A(-,\, ?)$**.** Let A be a ring, and M an A-module. If $T := - \otimes_A M$, then for every i

(*) $$\mathrm{Tor}_i^A(-,\, M) := L_i T.$$

Clearly if F is a flat A-module, then for every A-module D,

(**) $$\mathrm{Tor}_i^A(D,\, F) = 0 \qquad i \geq 1.$$

Recall that a flat module D is a module such that $(-) \otimes_A D$ is an exact functor. Suppose $D = D' \oplus D''$. Then $(-) \otimes_A D = (-) \otimes'_A \oplus (-) \otimes_A D''$ and hence $(-) \otimes_A D'$ and $(-) \otimes_A D''$ are exact if and only if $(-) \otimes_A D$ is. In other words, every direct summand of a flat module is flat.

Since projective modules are direct summands of free modules, and free modules are clearly flat, therefore, projective modules are flat. Using this we showed in class that

(***) $$\mathrm{Tor}_i^A(M,\, N) = \mathrm{Tor}_i^A(N,\, M).$$

Using (*), (**), and (***), we see that

$$\mathrm{Tor}_i^A(F,\, M) = 0 \qquad i \geq 1$$

for all A-modules M. In particular, if

$$0 \to U \to V \to F \to 0$$

is an exact sequence of A-modules with F a flat A-modules, then

$$0 \to U \otimes_A M \to V \otimes_A M \to F \otimes_A M \to 0$$

is exact for every A-module M.

The bifunctor $\mathrm{Ext}_A^i(-,\, ?)$**.** Let M and N be A-modules (A as above). Temporarily denote the i^{th} right derived functors of the two functors $\mathrm{Hom}_A(M,\, -)$ and $\mathrm{Hom}_A(-,\, N)$ by $\mathrm{Ext}_A^i(M,\, -)$ and $\mathrm{ext}_A^i(-,\, N)$ respectively. From class, we have

$$\mathrm{Ext}_A^i(M\, N) \xrightarrow{\sim} \mathrm{ext}_A^i(M,\, N)$$

for every M and N and every i. In fact this is a bifunctorial isomorphism, as is easy to check. We identify the two δ-functors, and the notation $\mathrm{Ext}_A^i(M,\, N)$ is the preferred one.

Cohomology of sheaves. Let X be a topological space, and $T = \Gamma(X, -) \colon \mathscr{S}h_X \to \mathcal{A}b$ the global sections functor, $\mathcal{A}b$ being the category of abelian groups. Then for every i, R^iT is denoted $\mathrm{H}^i(X -)$. For a sheaf $\mathscr{F}$, the group

$$\mathrm{H}^i(X, \mathscr{F})$$

is called the i^{th} *cohomology group* of the sheaf $\mathscr{F}$.

Ext groups in an abelian category $\mathscr{A}$. There is yet another notion of the Ext groups. Let $\mathscr{A}$ be an abelian category, and A, B two objects in $\mathscr{A}$. We set

$$\mathrm{Ext}^0_{\mathscr{A}}(A, B) := \mathrm{Hom}_{\mathscr{A}}(A, B).$$

For $i \geq 1$, $\mathrm{Ext}^i_{\mathscr{A}}(A, B)$ is the set of equivalence classes of exact sequences

$$(\dagger) \qquad 0 \to B \to D^1 \to \cdots \to D^i \to A \to 0.$$

We set $D^0 = B$ and $D^{i+1} = A$. Note that the sequence begins at B and ends at A! Two such sequences $D^\bullet$ and $E^\bullet$ are considered equivalent if there is an isomorphism of complexes $\varphi \colon D^\bullet \xrightarrow{\sim} E^\bullet$ with $\varphi^0 = 1_B$ and $\varphi^{i+1} = 1_A$. If $\mathscr{A}$ has enough projectives, then one cab show that $\mathrm{Ext}^i_{\mathscr{A}}(A, B)$ is computed by taking a projective resolution $P^\bullet \to A$ of A, and then calculating the cohomology of $\mathrm{Hom}^\bullet_{\mathscr{A}}(P^\bullet, B)$. In other words, in this case $\mathrm{Ext}^i_{\mathscr{A}}(-, B)$ is the derived functor of $\mathrm{Hom}_{\mathscr{A}}(-, B)$. Similarly, if $\mathscr{A}$ has enough injectives, $\mathrm{Ext}^i_{\mathscr{A}}(A, -)$ is the derived functor of $\mathrm{Hom}_{\mathscr{A}}(A, -)$. In the event $\mathscr{A}$ has both enough injectives and enough projectives, then by the technique used in class, clearly the last two definitions of $\mathrm{Ext}^i_{\mathscr{A}}(A, B)$ coincide and are groups. If any of the latter two are defined then they agree with the definition involving exact sequences. To give some idea of how the process works, suppose $\mathscr{A}$ has enough injectives. Let $B \to I^\bullet$ be an injective resolution of B. Since $(\dagger)$ is an exact sequence it gives rise to a resolution of B, and hence we have a map of complexes, unique up to homotopy, $f \colon C^\bullet \to I^\bullet$ where $C^\bullet$ is

$$0 \to D^1 \to \cdots \to D^i \to D^{i+1} \to 0$$

with D^1 in the zeroth spot (so $C^i = D^{i+1}$ for $i \geq 0$, and $C^i = 0$ for $i < 0$). We thus have a map $f^i \colon A \to I^i$. This gives an element of $\mathrm{Hom}^i_{\mathscr{A}}(A, I^\bullet)$. One checks it is a co-cycle (i.e., it is in the kernel of the coboundary map of $\mathrm{Hom}^\bullet_{\mathscr{A}}(A, I^\bullet)$) and hence gives an element in the i^{th} cohomology of $\mathrm{Hom}^\bullet_{\mathscr{A}}(A, I^\bullet)$. The reverse process is more difficult to describe, and we won't do it now.

The Abstract DeRham's Theorem

Throughout this section $T \colon \mathscr{A} \to \mathscr{B}$ is an additive functor, with $\mathscr{A}$ possessing enough injectives.

Definition 2. An object A of $\mathscr{A}$ is said to be T-*acyclic* if $R^iT(A) = 0$ for $i \geq 1$.

Lemma 1. *Suppose T is* left exact *and $N^\bullet$ a* bounded below *complex of T-acyclic objects in $\mathscr{A}$. Then the complex $T(N^\bullet)$ of objects in $\mathscr{B}$ is exact.*

Proof. It is enough to prove the statement under the assumption that $N^i = 0$ for $i < 0$. Let $C = \mathrm{coker}\,(N^0 \to N^1)$. The exact sequence

$$0 \to N^0 \to N^1 \to C \to 0$$

gives rise to a long exact sequence involving $R^iT(N^0)$, $R^iT(N^1)$ and $R^iT(C)$. Since N^0 and N^1 are T-acyclic, it follows easily from this long exact sequence that $R^iT(C) = 0$ for $i \geq 1$, whence C is also T-acyclic. A similar argument also shows that the sequence

$$(*) \qquad 0 \to T(N^0) \to T(N^1) \to T(C) \to 0$$

is exact, for $R^1T(N^0) = 0$. Let $R^\bullet$ be the complex with $R^i = 0$, $i < 1$, $R^i = N^i$ for $i \geq 2$, and $R^1 = C$. The $R^\bullet$ is a bounded below complex of T-acyclics, and $R^\bullet$ "begins" from degree 1. Moreover $R^\bullet$ is exact. Let $P^\bullet$ be the complex $0 \to N^0 \xrightarrow{1_{N_0}} N_0 \to 0$ with the two copies of N^0 placed in the 0^{th} and 1^{st} places. Then $P^\bullet$ is also an exact complex of T-acyclics. We have a short exact sequence of complexes

$$0 \to P^\bullet \to N^\bullet \to R^\bullet \to 0$$

which is perhaps better understood if displayed as the commutative diagram with exact rows and exact columns below:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & 0 & \to & N^p & \xrightarrow{=} & N^p & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & 0 & \to & N^2 & \xrightarrow{=} & N^2 & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & N^0 & \to & N^1 & \to & C & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & N^0 & \xrightarrow{=} & N^0 & \to & 0 & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & & & & & & & \\
 & & P^\bullet & & N^\bullet & & R^\bullet & &
\end{array}$$

One checks easily that for every i, the sequence

$$(\dagger) \qquad 0 \to T(P^i) \to T(N^i) \to T(R^i) \to 0$$

is exact. The only i which needs to be looked at carefully is $i = 1$, and here (*) does the trick. By (†), we have an exact sequence of complexes

$$0 \to T(P^\bullet) \to T(N^\bullet) \to T(R^\bullet) \to 0.$$

Since $T(P^\bullet)$ is clearly exact (identity morphisms are transformed to identity morphisms by functors), it follows that

$$(\ddagger) \qquad \mathrm{H}^i(T(N^\bullet)) \xrightarrow{\sim} \mathrm{H}^i(T(R^\bullet))$$

for every i. In particular, $\mathrm{H}^0(T(N^\bullet)) = 0$, something we can also deduce from the left exactness of T.

Fix n. Suppose $\mathrm{H}^i(T(K^\bullet)) = 0$ for all $i \leq n$ and all $K^\bullet$ which are exact, consisting of T-acyclics, and such that $K^i = 0$ for $i < 0$. Then using standard properties of cohomologies of translations of complexes we have:

$$\begin{aligned} \mathrm{H}^{n+1}(T(R^\bullet)) &= \mathrm{H}^n(T(R^\bullet[1])) \\ &= 0 \end{aligned}$$

for, $R^\bullet[1]$ satisfies the hypotheses on $K^\bullet$ above. Applying this to (‡) with $i = n+1$, we see that $\mathrm{H}^{n+1}(T(N^\bullet)) = 0$. □

Definition 3. A resolution $A \to F^\bullet$ of an object A in $\mathscr{A}$ is called a *T-acyclic resolution* if every F^i is T-acyclic.

Remarks 1. *An injective resolution is always a T-acyclic resolution, since injectives are T- acyclic for every additive functor T.*

The following theorem can be viewed as an abstract form of DeRham's theorem.

Theorem 5. *Suppose T is* left exact *and we have a T-acyclic resolution $A \to F^\bullet$ of an object A of $\mathscr{A}$. Then for every integer i we have a canonical isomorphism*

$$\mathrm{H}^i(T(F^\bullet)) \xrightarrow{\sim} R^iT(A).$$

In greater detail, let $A \to I^\bullet$ be an injective resolution of A and

$$\varphi\colon F^\bullet \to I^\bullet$$

the homotopy unique map of complexes lifting the identity map on A. Then for every integer i map

$$\mathrm{H}^i(T(\varphi))\colon \mathrm{H}^i(T(F^\bullet)) \to \mathrm{H}^i(T(I^\bullet)) =: R^iT(A)$$

is an isomorphism.

Proof. Let $\varphi\colon F^\bullet \to I^\bullet$ be as in the statement of the theorem. Then φ is a quasi-isomorphism. The mapping cone $C^\bullet_\varphi$ of φ is therefore exact and certainly bounded below. Since $C^i_\varphi = F^i \oplus I^{i+1}$, it is evident that C^i_φ is T-acyclic. Since T is additive we obviously have

$$T(C^\bullet_\varphi) = C^\bullet_{T(\varphi)}.$$

By the previous lemma, the left side of the above is exact, whence so is the right side. We are clearly done. □

Here is an obvious generalisation of the above:

Theorem 6. *Let $\psi\colon D^\bullet \to E^\bullet$ be a quasi-isomorphism of bounded below T-acyclic complexes. Then $T(\psi)\colon T(D^\bullet) \to T(E^\bullet)$ is a quasi-isomorphism.*

Proof. Same as the proof of the previous theorem. □