NOTES 1

Throughout, A is a ring. Here are some basic facts, notations, and defintions that will help you in understand the HW problems.

Basic notions

1) For an element $a \in A$, the *centralizer* of a, denoted $C(a)$ is the set of elements in A which commute with a, i.e.,

$$C(a) := \{x \in A \mid ax = xa\}.$$

For a subset S of A, we define the centralizer $C(S)$ of S to be the subset of A given by

$$C(S) := \bigcap_{a \in S} C(a).$$

2) If $\mathfrak{a}$ is an ideal of A, and M is an A-module, then $\mathfrak{a}M$ is defined to be the submodule generated by the set $\{ax \mid a \in \mathfrak{a}, x \in M\}$. Note that from what we saw in class, this means

$$\mathfrak{a}M = \{a_1x_1 + \cdots + a_nx_n \mid a_i \in \mathfrak{a}, x_i \in M, i = 1, \ldots, n\}.$$

3) Let M be an A-module and x an element of M. We have a natural surjective map $\mu_x \colon A \to Ax$ given by $a \mapsto ax$. The *annhilator of* x, denoted $\operatorname{ann}(x)$ is defined as the kernel of μ_x. Note that $\operatorname{ann}(x)$ is a left ideal and that

$$\operatorname{ann}(x) = \{a \in A \mid ax = 0\}.$$

Note that $x \in M_{\mathrm{tor}}$ if and only if $\operatorname{ann}(x) \neq 0$. Equivalently, $x \notin M_{\mathrm{tor}}$ if and only if $\operatorname{ann}(x) = 0$. We have an isomorphism

$$A/\operatorname{ann}(x) \xrightarrow{\sim} Ax.$$

Direct sums and products

Direct product of sets. Recall that if $(S_\alpha \mid \alpha \in I)$ is a family of sets[1] indexed by the indexing set I, then the set-theoretic product $\prod_{\alpha \in I} S_\alpha$ makes sense. It consists of families of elements $(s_\alpha \mid s_\alpha \in S_\alpha)$. For brevity we often write $(S_\alpha)_{\alpha \in I}$ and (S_α) to indicate a family. A similar convention is in operation for families of elements, so that $(s_\alpha) \in \prod_{\alpha \in I} \in S_\alpha$ is a short form for $(s_\alpha \mid s_\alpha \in S_\alpha) \in \prod_{\alpha \in I} \in S_\alpha$. Variants of this notation (all of which we will use) include $(s_\alpha)_\alpha$, $(s_\alpha)_{\alpha \in I}$, etc. It is standard to identify $(s_\alpha) \in \prod_\alpha S_\alpha$ with maps $s \colon I \to \cup_{\alpha \in I} S_\alpha$ such that the image of $\alpha \in I$ lies in S_α. The only point is that in this notation, the argument α is a suffix, in other words one writes s_α instead of the more conventional $s(\alpha)$. All of this should be familiar to you from the definition of a sequence of real or complex numbers used in

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[1] Informally, the difference between a family and a collection (i.e., a set) is that for sets, the collection of objects $\{A_\alpha \mid A_\alpha = A, \alpha \in I\} = \{A\}$, whereas in a family repetitions indexed by different elements are not collapsed, in other words$(A_\alpha \mid A_\alpha = A, \alpha \in I) \neq (A)$ unless the indexing set I consists of one element.

analysis. Note that if I is a finite set, say $I = \{1, \dots, n\}$ we make the identification $(s_\alpha) = (s_1, \dots, s_n)$, and often write $S_1 \times \dots S_n$ or $\prod_{i=1}^n S_i$ for the product of sets.

Direct product of modules. Let $(M_\alpha \mid \alpha \in I)$ be a family of modules. The *direct product* of this collection is the set $\prod_{\alpha \in I} M_\alpha$ with the addition law $(x_\alpha) + (y_\alpha) = (x_\alpha + y_\alpha)$ and with scalar multiplication $a(x_\alpha) = (ax_\alpha)$. We denote the direct product by the same symbol we used for the product of the underlying sets, namely the symbol $\prod_{\alpha \in I} M_\alpha$ (or just $\prod_\alpha M_\alpha$). For $\beta \in I$, we have the projection map $\pi_\beta \colon \prod_\alpha M_\alpha \to M_\beta$, namely $(x_\alpha) \mapsto x_\beta$. From the definition of the module structure on $\prod_\alpha M_\alpha$, it is clear that π_β is a module homomorphism. The direct product, together with the family of projections (π_α) has the following property. Let N be a module and suppose we have module homomorphisms, $\varphi_\alpha \colon N \to M_\alpha$, one for each $\alpha \in I$. Then there is a unique homomorphism of modules

$$\varphi \colon N \to \prod_\alpha M_\alpha$$

such that $\pi_\alpha \circ \varphi = \varphi_\alpha$, $\alpha \in I$. The map φ is the map $x \mapsto (\varphi_\alpha(x))$, $x \in N$. For the direct product of two modules, M_1 and M_2, this universal property can be expressed diagramatically as follows. Suppose we have a diagram as below. Then there is one and only one way to fill the dotted arrow to make the diagram commute.

$$\begin{tikzcd}
 & & M_1 \\
N \arrow[urr, bend left, "\varphi_1"] \arrow[drr, bend right, "\varphi_2"'] \arrow[r, dotted, "\exists ! \varphi"] & M_1 \times M_2 \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & & M_2
\end{tikzcd}$$

Direct sum of modules. The *direct sum of a family of modules* $(M_\alpha)_{\alpha \in I}$ is

$$\bigoplus_{\alpha \in I} M_\alpha = \{(x_\alpha) \in \prod_\alpha M_\alpha \mid x_\alpha = 0 \text{ for all but a finite number } \alpha \in I\}.$$

One checks that $\bigoplus_\alpha M_\alpha$ is a submodule of $\prod_\alpha M_\alpha$. For a finite number of modules, say $M_1, \dots, M_n$, we often write $\bigoplus_{i=1}^n M_i$ or $M_1 \oplus \dots \oplus M_n$ for the direct sum. We have injective homomorphisms of modules (monomorphisms) $i_\beta \colon M_\alpha \to \bigoplus_\alpha M_\alpha$, one for each $\beta \in I$, given by $\pi_\alpha(i_\beta(x)) = x$ if $\alpha = \beta$ and is 0 otherwise. Note that this defines i_β, for it describes $i_\beta(x)$ as a map $I \to \cup_\alpha M_\alpha$, namely as $\alpha \mapsto \delta_{\alpha,\beta} x$, where $\delta_{\alpha,\beta}$ is the Kronecker delta. It is easy to see that i_β is an injective map and a module homomorphism. The direct sum also has a universal property which is in some sense "dual" to the universal property of direct products, namely, if N is a module and we have module homomorphisms $f_\alpha \colon M_\alpha \to N$, one for each $\alpha \in I$, then there is a unique homomorphism of modules $f \colon \bigoplus_\alpha M_\alpha \to N$ such that $f_\alpha = f \circ i_\alpha$. For two modules M_1 and M_2, this universal property is expressed in terms of the diagram below, namely there is a unique map which fills the dotted arrow to make the diagram commute.

$$\begin{tikzcd}
M_1 \arrow[dr, "i"'] \arrow[drr, bend left, "f_1"] & & \\
& M_1 \oplus M_2 \arrow[r, dotted, "\exists ! f"] & N \\
M_2 \arrow[ur, "j"] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

Sums and Internal direct sums. Recall that if if N is an A-module, and M_α, $\alpha \in I$ are submodules of N then $\sum_\alpha M_\alpha$ is the submodule of N generated by $\cup_\alpha M_\alpha$. Let $M = \sum_\alpha M_\alpha$. Then each M_α is a submodule of M. We have inclusion maps $j_\alpha \colon M_\alpha \to M$ which are A-module homomorphisms. By the universal property of direct sums, we have a unique map

$$j \colon \bigoplus_\alpha M_\alpha \to \sum_\alpha M_\alpha = M$$

such that $j \circ i_\alpha = j_\alpha$ for every α. The map j is clearly surjective. Indeed j is also described by $(x_\alpha) \mapsto \sum_\alpha x_\alpha$, and using this description surjectivity is clear. In the event this is an isomorphism, we do not distinguish between $\bigoplus_\alpha M_\alpha$ and $\sum_\alpha M_\alpha$ and write

$$\bigoplus_\alpha M_\alpha = \sum_\alpha M_\alpha.$$

Free Modules

Let M be an A-module. A subset S of M is said to be *linearly independent* if whenever $\sum_\alpha a_\alpha s_\alpha = 0$, we have $a_\alpha = 0$ for all α. A *family* of elements $(s_\alpha \mid \alpha \in I)$ in M is said to be *linearly independent* if the same condition holds, namely if $\sum_\alpha s_\alpha = 0$ then every s_α is zero. Note that if A is a field, and M is a non-zero vector space, and $x \in M$ is a non-zero element, then the set $S = \{x, x, x\}$ is linearly independent (since $S = \{x\}$), but the family (x, x, x) is not linearly independent. A set (resp. family) which is not linearly independent is said to be *linearly dependent*. Note that if a family of elements $(s_\alpha \mid \alpha \in I)$is linearly independent, it necessarily consists of distinct elements, and in this one instance, we have no confusion if we identify the family $(s_\alpha \mid \alpha \in I)$ with the set $\{s_\alpha \mid \alpha \in I\}$.

If S is linearly independent and generates M it is said to be a *basis* of M. Suppose $S = \{s_\alpha \mid \alpha \in I\}$ is a basis of M. Then, as for vector spaces, every element x of M has a *unique* representation as $x = \sum_\alpha a_\alpha s_\alpha$ where $a_\alpha \in A$. Similarly one defines what it means for a family $(s_\alpha \mid \alpha \in I)$ to be a basis, and in view of the remark in the last sentence of the last paragraph, there is no confusion in switching from families to sets (or vice versa) when talking about bases.

If (s_α) is linearly independent, then $a s_\alpha = 0$ if and only if $s_\alpha = 0$. Therefore no member of a linearly independent set of family can be a torsion element. In other words

$$\operatorname{ann}(s_\alpha) = 0$$

for every α.

Not every module has a basis. Consider the following example. Let $A = \mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$. Since every element of M is a torsion element (take $a = 2$, then $ax = 0$

for all $x \in M$), it is clear that M cannot have a linearly independent set, and so M cannot have a basis.

Definition. A module M is said to be *free* if either $M = 0$ or if it has a basis.