COKERNELS

This note answers the following question:

Why is $\ker(\widetilde{Z}^n \to Z^{n+1}) = \mathrm{H}^n(X^\bullet)$? Here $X^\bullet$ is a complex in an exact category $\mathscr{C}$, and for $p \in \mathbb{Z}$, $Z^p = \ker(Z^p \to Z^{p+1})$ and $\widetilde{Z}^p = \operatorname{coker}(X^{p-1} \to X^p)$ and the map $\widetilde{Z}^n \to Z^{n+1}$ is the natural map arising from the definition of kernels and cokernels. The note also answers another related question (see diagram at the end of the note).

Suppose $\mathscr{C}$ is an exact category and $A \xrightarrow{i} B \xrightarrow{j} C$ are two monomorphisms. Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & = & A & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle ji} & & \downarrow & & \\ 0 & \longrightarrow & B & \xrightarrow[j]{} & C & \longrightarrow & \operatorname{coker}(j) & \longrightarrow & 0 \end{array}$$

By the Snake Lemma as well as 2 (c) of HW-5 we get an exact sequence

$$(*) \qquad 0 \to \operatorname{coker}(i) \to \operatorname{coker}(ji) \to \operatorname{coker}(j) \to 0.$$

Moreover the diagram

$$(\dagger) \qquad \begin{array}{ccc} B & \xhookrightarrow{\;j\;} & C \\ \downarrow & & \downarrow \\ \operatorname{coker}(i) & \hookrightarrow & \operatorname{coker}(ji) \end{array}$$

commutes.

Apply this the following special case. Let $X^\bullet$ be a complex and n an integer. Let $A = \mathbf{im}(\partial^{n-1})$, $B = Z^n$ and $C = X^n$. Then $(*)$ gives us an exact sequence

$$0 \to \mathrm{H}^n(X^\bullet) \to \widetilde{Z}^n \to \mathbf{im}(\partial^n) \to 0.$$

Thus $\mathrm{H}^n(X^\bullet) = \ker(\widetilde{Z}^n \to \mathbf{im}(\partial^n)) = \ker(\widetilde{Z}^n \to Z^{n+1})$. Further, diagram $(\dagger)$ gives us

$$\begin{array}{ccc} Z^n & \xhookrightarrow{\;j\;} & X^n \\ \downarrow & & \downarrow \\ \mathrm{H}^n(X^\bullet) & \hookrightarrow & \widetilde{Z}^n \end{array}$$