HW3

We work throughout in an abelian category $\mathscr{A}$. Fix a complex $(\mathcal{K}^\bullet,\, d)$ which is the total complex of an *anti-commuting* double complex $K^{\bullet\bullet} = (K,\, d_1,\, d_2)$ with the property that $K^{\bullet\bullet}$ is bounded on the left and below by p_0 and q_0 respectively. Recall that under the last hypothesis, $\mathcal{K}^\bullet = {}'\mathrm{Tot}^\bullet K^{\bullet\bullet}$ exists without assuming $\mathscr{A}$ has countable direct sums.

(1) Let $A^{\bullet\bullet} = (A,\, \partial_1,\, \partial_2)$ and $D^{\bullet\bullet} = (D,\, \delta_1,\, \delta_2)$ be data given by

$$A^{p,q} = D^{p,q} = K^{p,q}$$

and whose partial coboundaries are given by:

$$\partial_1^{p,q} = d_1^{p,q} \qquad\qquad \partial_2^{p,q} = (-1)^p d_2^{p,q}$$
$$\delta_1^{p,q} = (-1)^q d_1^{p,q} \qquad\qquad \delta_2^{p,q} = d_2^{p,q}.$$

(a) Show that $(\mathrm{Tot}^\bullet A^{\bullet\bullet},\, \partial) = (\mathcal{K}^\bullet,\, d)$.
(b) Show that $\mathrm{Tot}^\bullet D^{\bullet\bullet}$ is isomorphic to $(\mathcal{K}^\bullet,\, d)$.

(2) Suppose $\mathscr{A}$ is the category of left modules over a ring. Show by *chasing elements* that if the columns of $K^{\bullet\bullet}$ are exact, then $\mathcal{K}^\bullet$ is exact.

(3) Do the above problem without the assumption made on $\mathscr{A}$ in that problem.

Recall that each row and each column of $K^{\bullet\bullet}$ is a complex. For a fixed q, let $R_q^\bullet$ be the q^{th} row, and for a fixed p, let $C_p^\bullet$ be the p^{th} column. In somewhat greater detail the n^{th} term of $R_q^\bullet$ is $K^{n-q,q}$ (and *not* $K^{n,q}$!). We point out that $R_q^\bullet$ is also the q^{th} row of the standard double complex $A^{\bullet\bullet}$ of Problem 1 and $C_p^\bullet$ is almost the p^{th} column of $D^{\bullet\bullet}$ of the same problem—the modification needed is that coboundaries of the new column are obtained from the old by multiplying by $(-1)^p$. To lighten notation, write $R^\bullet$ for the *bottom most* row i.e., $R^\bullet = R_{q_0}^\bullet$, and write $Z^\bullet$ for the sub-complex of $R^\bullet$ defined by $Z^n = \ker(d_2^{n-q_0,q_0})$.

(4) Show that the natural inclusion $Z^\bullet \hookrightarrow \mathcal{K}^\bullet$ given by the composite $Z^n \hookrightarrow A^{n-q_0,q_0} = K^{n-q_0,q_0} \hookrightarrow \mathcal{K}^n$ is a map of complexes.

Problem 5 is on the next page.

Let $Z^\bullet \hookrightarrow R^\bullet$ be as above. Let

$$\varphi\colon C^\bullet \to Z^\bullet$$

be a map of complexes, where as before. Define $\widetilde{A}^{\bullet\bullet} = (\widetilde{A}, \widetilde{\partial}_1, \widetilde{\partial}_2)$ as follows. The objects at the $(p,q)^{\text{th}}$ spot are: $\widetilde{A}^{p,q} = A^{p,q}$ for $q \neq q_0 - 1$, $\widetilde{A}^{p,q_0-1} = C^{p+q_0}$. The partial coboundaries given are described thus. For $q \neq q_0 - 1$ set $\widetilde{\partial}_1^{p,q} = \partial_1^{p,q}$, and $\widetilde{\partial}_2^{p,q} = \partial_2^{p,q}$. When $q = q_0 - 1$ set $\widetilde{\partial}_1^{p,q_0-1} = \partial_C^{p+q_0}$, $\widetilde{\partial}_2^{p,q_0-1} = \varphi^{p+q_0}$, where ∂_C is the coboundary of $C^\bullet$.

(5) (a) Check that $\widetilde{A}^{\bullet\bullet}$ is indeed a double complex.

(b) Composing $\varphi\colon C^\bullet \to Z^\bullet$ with the map in Problem 4, we get a map of complexes $\psi\colon C^\bullet \to \mathcal{K}^\bullet$. Show that ψ is a quasi-isomorphism if and only if $\operatorname{Tot}^\bullet \widetilde{A}^{\bullet\bullet}$ is exact.

Remark. Note that if we have a complex $C^\bullet$ and maps $\varphi^n\colon C^n \to A^{n-q_0,q_0} = K^{n-q_0,q_0}$, $n \in \mathbb{Z}$, such that the resulting maps $C^n \to \mathcal{K}^n$ give us a map of complexes, $C^\bullet \to \mathcal{K}^\bullet$, then φ^n must factor through $Z^n \hookrightarrow A^{n-q_0,q_0}$, i.e., the map of complexes $C^\bullet \to \mathcal{K}^\bullet$ must factor through $Z^\bullet \hookrightarrow \mathcal{K}^\bullet$.